JOURNAL OF APPLIED PHYSICS

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In This Issue

		Page
Current Trends in the American Physical Society		437
National Headquarters Building for Physics		442
Some Physical Procedures Used in Ceramic Research	G. A. Bole and G. A. Loomis	443
Powdered Iron Cores	Alfred Crossley	451
Frictional Phenomena. XV	Andrew Gemant	456

Contributed Original Research

Approximate Mathematical Methods of Applied Physics as Exemplified by Application to Saint-Venant's Torsion Problem	Thomas James Higgins	469
Distribution of Light Intensity within a Scattering Medium	Robert R. Newton	481
Applications of the Operational Calculus to the Theory of Structures	Louis A. Pipes	486
Radioactive Mercury in Mercury-Vapor Measurements	John W. Irvine, Jr., and Clark Goodman	496
Errata: Flow and Recovery Properties of Viscose Rayon Yarn	Jack J. Press	464

In Every Issue

Calendar of Meetings	455
Résumé of Recent Research	465
New Books	465
Here and There	466
Recent Applications of Physics	467
Innovations in Instruments	468
Index to Advertisers	xviii

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Current Trends in the American Physical Society

Recently, there has been considerable agitation in the American Physical Society for greater participation in the affairs of the Society by physicists in industry. At the suggestion of the Council of the Society there is given here a brief review of the discussion of the past few months. There follows first a note by Dr. Karl K. Darrow, Secretary of the Society, giving the background of this discussion and explaining the Council's attitude regarding the formation of divisions within the Society. Then follow excerpts from letters written by members of the Society which indicate the general trend of the discussion. Since the letters were not sent in for publication no names are attached to the excerpts. If space permits additional comments will be printed as they are received.—The Editor.

In recent years there has been a great expansion of activity in the science of physics; the number of those who cultivate it has grown considerably, and the diversity of their specialties has increased correspondingly. Moreover a steadily rising proportion of physicists has been employed by industry, a lesser proportion (though a rising absolute number) by universities. These trends seem likely to go on for years to come, perhaps indeed at an ascending rate. It is reasonable to inquire whether the structure and the procedures of the American Physical Society, which have altered but little in the last thirty years, may now be modified with advantage.

This question has been considered by the Council of the Society from time to time for several years. In 1931 the following article was inserted into the Constitution of the Society to provide a mechanism for the formation of Divisions within the Society:

"Article IX. 1. The Council may, upon petition by members of the Society, form a Division within the Society charged with the advancement and diffusion of the knowledge of a specified subject or subjects in physics. 2. Each Division shall elect an Executive Committee, the Chairman of which shall report its activities and needs to the Council. 3. Any Division may be dissolved at the discretion of the Council."

At each of the Council's last few meetings, including a special meeting held on June fifth, there was discussion of the opportunity afforded by this Article and the ways of profiting by it. These recent discussions were stimulated by several letters from members inquiring whether the Society could give a greater place and role in its activities to physicists in industry and in various branches of applied physics. Fortunately, in the spring of 1943, advantage was finally taken of Article IX by a group organized by Dr. L. Marton of Stanford University, who wrote and circulated a petition for the establishment of a division of Electron and Ion Optics.

Signed by forty of the Fellows and Members of the Society, this petition was accepted by the Council on May first, and a committee was appointed to prepare a scheme for getting the new Division under way. Further details need not be given in this place, for the entire Society has already received a description (published in the August *Journal of Applied Physics*) of the plans for the new division and of the procedure for enrolling therein. This first division will give to the Council and the Society a welcome chance of observing how a division works and deducing how future divisions should be administered.

The Council will welcome suggestions and propositions for other divisions, each to be limited in scope to a particular field or fields of physics. The limitation should be reasonably clear-cut; but no one need apprehend that by defining the scope of one division he is cutting himself or his colleagues off from other fields, for it is intended that any member of the Society may enroll in any or all divisions. It is realized that in the present conditions, very many physicists are entirely absorbed in urgent work and cannot give any time to the activities of a division. The proverb "In time of war, prepare for peace" is nonetheless valid here. We shall be designing an adequate framework, in which the work of the Society can develop as soon as happier times arrive.

Letters have been received conveying the idea of a "Division of Industrial Physics." The Council is disinclined to favor this idea, which incidentally appears to be precluded by the language of Article IX. One of the things most greatly to be desired is a unification of the physicists called "academic" and the physicists called "industrial." It is important to avert the danger of a lack or loss of interest by either group in the problems and the enterprises of the other. Now, the establishment of a "Division of Industrial Physics" would work in exactly the opposite direction. That way lies the peril of forming two contrasting groups, whose aims and ambitions ought to be alike but would infallibly deviate more and more as the years go on.

—KARL K. DARROW

Excerpts from Letters of Members of the American Physical Society:

Need for Greater Unification in Program

I have heard it said for a long time that there is dissatisfaction on the part of industrial physicists with the conduct of the affairs of the society, but I have yet to meet such individuals personally. I don't mean to imply that they don't exist, only that I have no personal experience with them.

Personally, I think the dissatisfaction stems largely from the fact that industrial physics covers such a wide range of topics that it is hard even among the industrial physicists to get together much of an audience for any highly specialized paper analogous to the usual contributed papers of our program. That is not the fault of the society, but simply a statement of the basic fact and I do not see much that the society can do to correct it. I think it should always be emphasized that the society welcomes papers from industrial physicists and welcomes their taking the initiative in setting up special symposia on topics of especial interest to them.

I feel very strongly opposed to any movement to have a division of industrial physics or any other such plan which would act to emphasize any division of interests of those physicists who are employed in industry and those employed in academic institutions. We have too much separation now, so what we need is means to bring them together instead of means to take them apart.

It seems to me that the society has done well in fostering so successfully the *Journal of Applied Physics*. It seems that the people who are thinking of a separatist movement should be encouraged to take the initiative in planning program material for the American Physical Society.

Reciprocal Benefit Derived from Contact; Sociability Needed

Personally, I have never felt that the Society discriminates against industrial physics and those who practice it. . . . The chief aim of the American Physical Society has been, and I believe should continue to be, that of promoting the development of physics by bringing physicists *together*. Every physicist knows that the various branches of physics are parts of the science as a whole and fundamentally are mutually dependent. At this time, more than ever before, the Society should use its influence to discourage the practice of the various branches of physics by groups of "isolationists."

It is obvious that I do not believe there should be a "Division of Industrial Physics." All physicists learn their fundamentals from the academic physicist. The two groups should have much in common. The academic physicist should be much interested in the progress, success, and welfare of the industrial physicist because the latter is his chief product. The best way to learn how, and what, to teach future physicists is to keep in close touch with those out in the industrial field. If the academic physicist is not directly interested in carrying his research to practical ends he should, and no doubt does, desire to produce men who are interested in applying the results of academic research to the development of things with which people can make life and the world more livable.

I suggest that the Society should make the meetings more like social gatherings and less like dry old class room meetings in which the "professor" is away from his home apparatus and with his "lesson" poorly organized for teaching purposes. It might be a good plan for the Society to require the presence of every speaker at several luncheons and dinners. Instead of one or two tables there should be one for each group of subjects on the program. The committee would place the speakers at the proper tables. Members planning on attending the meeting could select their tables in advance. This should bring the members in contact with the speakers and help to break the ice for the industrial physicists who are comparatively young and timid.

In exchange for the privilege of appearing before the Society the committee has a right to expect the assistance of the speakers in socializing the meetings. The academic group should be interested in cooperating, because if there is to be money for pure research there must be an active well satisfied group of industrial physicists working with them in an endeavor to make the results of pure research useful to society as a whole.

Revise Program Material and Its Presentation. Safeguard Professional Interests

. . . The lack of interest in American Physical Society affairs among industrial physicists is evidence that something is wrong . . . papers, both at the meetings and in publications, seem to be intended mainly for the experts and the average industrial physicist, who must be interested in many fields of physics, cannot be an expert in all. It seems that in all papers, even those that are largely mathematical, the results, if not the method, could be put in plain English.

In general, a good paper cannot be given to a heterogeneous audience in 10 minutes, and certainly all papers are not of equal importance. Any change would require that someone judge the relative importance of various papers. . . .

Modern physics has become largely a study of electrons and nuclei with less attention given to anything as large as a molecule. At least these subjects comprise most of the papers on recent programs. There are many other fields important to industry, for example, molecular physics and, to mention specific cases, molecular motions resulting in dipole loss and mechanical hysteresis. . . . Apparently many authors feel that certain types of physics papers will reach more interested people at other meetings. In subject matter, sectional American Physical Society programs seem to differ from those of national meetings and this may be significant.

Industrial physicists are just beginning to be generally important and after the war there will be many of them. There are organizations looking out for the welfare of other groups such as those representing engineers, chemists, and the medical profession in various ways. For example, salary surveys are made so that an engineer or chemist can have some idea whether he is in the proper pay group, and a watchful eye is kept on legislation which may effect them, such as for example, the proposed Kilgore Bill. These particular things may not be desirable, but the general program is, and no existing organization is doing these things for physicists.

Consider Industrialist's Interest in Program

I appreciate very much being given an opportunity to discuss the relationship of the American Physical Society to industrial physicists. I believe the chief difficulty in the situation is that the education of the physicist is not in accord with that demanded by the industrial laboratory, and this condition is the prime cause of the present situation. Physicists in the past have been trained primarily for teaching and institutional research. At the present time, and for a certain period in the future at least, industrial institutions and development research laboratories can absorb a large proportion of the physics graduates of this country.

More data are needed to indicate just what work the physicists of this country are now engaged in. The American Institute of Physics, as you are well aware, has compiled the information on the present manpower situation. I am not aware that the Physical Society has taken any steps to attempt to plan programs along the lines of the activities of the industrial members. I suspect its plan has been to "let nature take its course." I would suggest that the American Institute of Physics go further with its compilation of facts regarding its members and regarding the graduates in physics. I believe it will be found that the Physical Society now attracts to membership a smaller proportion of recent graduates in physics than formerly, say in 1910. If that is the case, you should take serious thought.

Papers from industrial groups are much harder to get than papers from pure research laboratories. In spite of the fact that some very valuable scientific work comes out of these industrial laboratories, the situation is that these men are more interested in the promotion of their activities

along industrial lines than they are in publication. Due to the present attitude in Washington towards patents and their present tendency to invalidate patents, it will be more and more difficult to secure papers from industrial laboratories and very few papers would ever be volunteered. A program of secrecy must from necessity be followed.

I have yet to attend a symposium which attempted to tie-in the industrialist and the pure scientist which I thought in any way adequate. The men simply do not get together. It is hardly possible for the industrialist to call to attention his vital problems, because the things he is most interested in and which are most alive are no doubt being worked on somewhat secretly by his own company or group. He mostly talks about the things which have already been done so that the scientist is left with the feeling that much progress has been made and that there is little that he can be expected to contribute to the art, yet the industrialist knows that is not the case. The educational institution is, in many cases, too much out of touch with the exact problems facing the industrialist so their attempts to cover the theoretical side of the subject also fall short.

Committees on (1) Governmental Relations and (2) Training

I suggest:

(1) The American Physical Society appoint a committee to prepare some form of action, possibly a session with invited speakers who will talk on certain phases of our Government which are of vital interest to the industrial physicists and their future such as:

(*a*) The present attitude to our patent laws and the attitude of our central government towards the patent system, which as you know is an attitude tending towards the destruction of the value of patents. The future of the industrial physicist is definitely tied up with a method of protecting his efforts so that he and his company may have compensation. If this protection should cease, there will be no point in training men in applied physics and engineering. I have seen nothing about any activities of our society which touch this subject at all. Yet, I believe this subject is of great interest to any professional physicist.

(*b*) The relationships of governmental laboratories and industrial laboratories. There is a bill, at the present time, in Washington which proposes to coordinate all types of research laboratories under governmental control. This you no doubt know of and I dare say that 95 percent of the American Physical Society, if they know of it at all, would be vitally opposed to such a bill, inasmuch as it definitely would remove the incentive motive from research.

(2) My second suggestion is that a committee be appointed to investigate and recommend courses of training planned for the development of industrial researchers and substituted for the present Ph.D. course in physics. This committee should obtain data regarding the fields of physics now being worked upon by industrial physicists and more definite information as to the work these men are doing to serve as a guide for this planning. I would recommend to the committee that they seriously consider the substitution of an additional degree, call it "Doctor of Applied Physics" or "Doctor of Physical Engineering," and the abolishment of the present type of Ph.D. thesis, substituting for that much more thorough training in experimental technique along the lines indicated by the summary of the report on the type of work that the physicists are now called upon to do. I advise this because I believe that very often the present experimental thesis does not develop research ability or experimental technique.

I believe that the installation of such courses and work will lead to publications and papers along the lines which tend to hold and attract industrial physicists to the American Physical Society. I think these changes in our educational program are sure to come and that the American Physical Society is the proper agency for the promotion of these or better ideas.

Experiment with Divisions on a Temporary Basis

Let it be understood that in going to the Divisional plan we are moving toward the formation of small groups which would work seriously through most of the duration of a meeting, adjourning only for the purpose of joining others in hearing addresses of general interest, arranged by the Secretary for the benefit of the whole membership. As soon as we do this, we gain several advantages. (1) We provide adequate time for real interchange of information between people with common interests. (2) We get away from the confusion of big meetings in big lecture halls. (3) We undoubtedly would get a much larger number of contributions because of the accelerating influence of Divisional Officers. Many a man would offer a paper on the current status of his work if he were assured of a small and friendly audience well acquainted with his field, when, on the other hand, he would hesitate to face a large crowd with the practical certainty that not one hearer in ten would have the slightest interest in what he was saying. Let us replace the artificial and unscientific meetings we have now by real working groups, held together by the strong bond of common interest. Only in this way can we recapture the delightful, unhurried atmosphere of earlier days.

We should establish Divisions in several fields of physics at once, with the clear understanding that the

life of each one would be two or three years, after which it would continue, if successful, or would be abandoned, if unsuccessful, with no prejudice against its reestablishment at a later time.

It is also important to realize that a very great body of academic physicists have turned to engineering physics during this conflict. It is a matter of common remark among these men that when they return to their regular places of employment they will go back with new interests and that many of them will continue to work on the practical problems of industry and of government. I do not believe this trend will weaken the field of pure physics; I believe that both pure and applied physics will be bigger and better after the war. Let us do what we can now to help the practical physicists, since most of us are practical physicists now; and let us do what we can to have a greater and stronger Society, commanding the firm allegiance of all physicists in the time of peace.

Greater Promotion, Less Patronization

I wish that you had addressed me as a "physicist in industry" on the subject "Increasing the share which physicists in industry take, etc." instead of as an industrial physicist.

I believe that the Society should not stress too often that physics is the parent science, and not attempt to reserve "physicist" as a term of any more distinction than "chemist," because the attempt will more likely fail. If the term "professional physicist" is to be reserved for research workers only, then we can look for trouble unless and until colleges begin awarding degrees in engineering or applied physics.

The Society will have to face the unpleasant (to some) fact that the use of the title Dr. as a label of distinction for physicists is a hindrance to full cooperation of the Society with engineering societies where many engineers of comparable training and experience do not have a special title of address. . . . Suppose we have in a laboratory group an able electronic engineer with an E.E. and some years of experience, with a group of assistants including a physicist with a doctor's degree and no experience. The test of personalities is apt to be rather severe.

To earn more money, it will be necessary to improve the value of the editorial matter in the Journal and in *Review of Scientific Instruments*, build up the circulation and advertising, hold conventions at which manufacturers pay for exhibit space, possibly even support a laboratory or two for a purpose of building up royalties on inventions.

To a large extent, the officers of the Society are elected for their accomplishments in research. The marked success of some of the engineering societies is due to the organizing ability and personality of a paid manager, sometimes called that and sometimes secretary. The development of more successful schemes for promoting local chapters, divorced from schools, frequently visited by one or another of the national officers, and thoroughly instructed and guided in their local management might help marvelously in the growth of the Society, besides helping in promoting contacts with physicists in industry.

The book [*Physics in Industry*] is very convincing to physicists, who did not need to be convinced if they have been keeping themselves informed, but it must be irritating to engineers, at many spots, to be told, for example, that physicists like to thoroughly understand the fundamentals, with the implication that engineers do not.

Greater Stress on Borderline Fields

Your letter regarding the share which industrial physicists take or should take in the activity of our Society affairs is of great interest to me.

There is a group here which is known as the "Physics Department." What we do would best be called fundamental work in chemical physics using infra-red and ultraviolet spectroscopy, x-ray spectroscopy, electron diffraction, electron microscopy, molecular ray technique, mass spectrography, precision calorimetry, radioactivity, all of which are, I should say, truly physical methods. We work in the fields of catalysis, reaction kinetics, lubrication, on analytical problems, etc. Are we chemists or are we physicists? Well—most of us are physicists by education and still like to be called physicists, although we are definitely working in the borderline field of chemical physics. We also like to work in a "Physics Department"; in fact, we are proud to be physicists and we feel that being a physicist should not bar us from a career in the oil industry or the chemical industry. . . . Physicists actually take an increasing share of responsibilities in industries which, off hand, do not appear to be physical industries such as Bell Telephone, Signal and Switch, G.E., and Westinghouse. If the chemical physicists' interest in the Physical Society (just to mention this borderline case) is to be preserved, the borderline must be stressed a good deal more than this has been done in the past.

I should like to suggest that the Physical Society take cognizance of this situation by encouraging symposia and papers in applied physics and borderline fields. The initiative should come from the officers of the Society, and I am sure that the applied physicists will gladly assume the laborious task of arranging the symposia, etc. I personally am strongly opposed to a division of applied physics within the Physical Society, but I would favor subject divisions such as the proposed division of Electron Optics. But most of all—and I feel very strongly on this point—the applied or industrial physicist should not have to be content or should not be content to publish a piece of apparatus in an instrument journal or a piece of spectral analysis in the Review both entirely detached from the important purpose which his branch of physics may play in his industry. We should strive for a more general pre-

sentation of the industrial physicist and his work within his industry in such a way that the presentation would appeal to physicists as well as to scientists or engineers on the other side of the borderline, and possibly to company research executives who in this way may learn what physicists could contribute or are contributing to their industry. Well-organized symposia, properly announced throughout the scientific literature, will undoubtedly be the best medium to achieve these ends.

Failure to see or appreciate the industrial viewpoint is well illustrated by the assumption that "The natural way is to divide physics into subjects, as for instance acoustics, thermionics, line spectra, etc." This I agree *is* the *natural* way, from the standpoint of teaching physics and also from the standpoint of learning physics or cataloguing physics as in Science Abstracts. However, it is *not* the only natural way of dividing the subject from other standpoints, including the standpoint of using physics.

The person using physics to improve a material or a process must bring parts of all these fields to bear on his particular problem, and he likes to discuss his results and methods in a group whose interests and viewpoints do not all form an orthogonal system to his own. The need for this close linking of the different "natural" divisions of physics and their applications to specific problems is well illustrated in the field of metallurgy, where physics and chemistry and engineering are all tied together in one group.

More Time, Space and Activity for Physicists in Industry

Regarding the best solution of this problem, I have no hard and fast convictions about what positive action should be taken. I do feel that merely not rejecting any articles submitted by industrial physicists is not enough and that holding symposia of invited papers is not enough. I think the solution will involve giving the industrial physicists more of a share in all of the activities of the Society (not only the meetings), and it will involve a recognition of the fact that publications by industrial physicists usually are the result of accumulating much data that cries out for publication rather than feeling that at least one publication a year is required for holding a job. One result of this is that ten minutes is not enough time for the presentation of any large amount of data, and they feel that it is not worthwhile to prepare such a paper. More flexibility in the time allowed for presenting research reports would help this situation.

I feel that the problem will not be solved until there are at least as many reports of physical research published in the *Journal of Applied Physics*, as there are published, for example, in *Industrial and Engineering Chemistry*. It is the publication of *The Journal of Chemical Physics* and the *Journal of Applied Physics* which today makes the American Physical Society worthwhile for many physicists, and the council and editors are to be congratulated on publishing these excellent magazines.

A National Headquarters Building for Physics

ON July 24th the Governing Board of the American Institute of Physics approved a resolution authorizing the Institute to organize an appeal for funds for the purchase of a building to be devoted to the interests of American Physics and Physicists. An exceptionally desirable property at 57 East 55th Street in Manhattan had already been located and a short-term option at a favorable price was secured. A Physics Building Fund Committee was appointed which promptly began a fund raising campaign by distributing letters and other material[1] presenting the project in considerable detail to physicists and their friends. Attention was called to the desirability of acquiring a headquarters building as the first step in a comprehensive program for the development for physics outlined by the War Policy Committee of the Institute.

The initial response to this presentation was quick and enthusiastic and by August 20th cash and pledges amounting to $31,816.75 had been received. On the strength of this assurance a contract was signed to purchase the property calling for a substantial down payment on October 1st, the balance of the purchase price to be paid in regular instalments during the following six months. To meet these contractual terms a continuing flow of subscriptions must be maintained until the goal of $75,000 is reached.

Physicists and their friends who have not had an opportunity to give thoughtful consideration to the merits of this progressive action are urged to do so.[2] Complete information may be obtained from the American Institute of Physics, 175 Fifth Avenue, New York 10, New York.

[1] Rev. Sci. Inst. **14**, 241–247 (Aug. 1943).
[2] See also pages xi and xii of this issue.

Some Physical Procedures Used in Ceramic Research*

BY G. A. BOLE AND G. A. LOOMIS

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MUCH engineering research procedure and its development into routine tests is an adaptation of fundamental physics and most of the research tools employed therein were originally conceived and developed by the physicist. This statement is particularly true of ceramic research, and especially so of that being carried on in the Engineering Experiment Station of The Ohio State University. The present paper covers some of these physical procedures as employed in the researches which have been or are now being carried on in the Station. Some of the physical procedures as adapted to ceramic research are not new while others are applications of quite recently developed physical apparatus and techniques. In describing these various techniques and their applications the detailed procedures need not be given herein since it may be assumed that the physicist is familiar with them.

DETERMINATION OF THERMAL EXPANSION

Volume changes in ceramic products, both clay products and glasses, during the heating and cooling cycle determine largely their resistance to rupture when heated. A glass baking dish must resist thermal shock; a glaze must fit the body which it covers. If the glaze is in high tension it crazes, if in high compression it shivers or dunts.

Fired ceramic systems usually consist of a crystalline and a glassy phase. The crystalline phase may be of one or more components. The characteristics of these phases so far as their reaction to thermal changes is concerned are recorded in the coefficient of expansion curve. These curves constitute significant information regarding the action of ceramic products over a given heat range.

The coefficient of thermal expansion is usually determined in one of two ways: (1) in a dilatometer or (2) in an interferometer.

A diagram of the dilatometer is shown in Fig. 1. The test specimen for this apparatus is a rod of the material under investigation about ½ inch in diameter and 8 inches long. This is heated in a fused quartz tube in an electrical resistance furnace at a rate of about 150°C per hour. The expansion of the specimen is measured by means of an Ames dial attached as shown to the top of

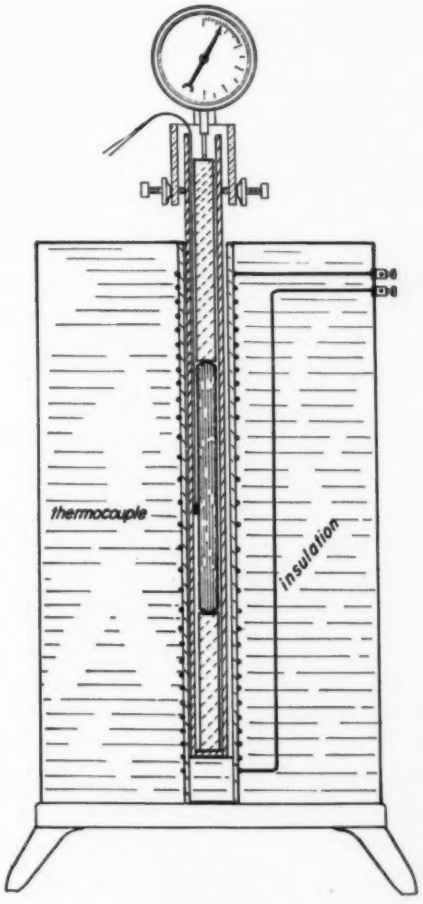

FIG. 1. Diagram of dilatometer apparatus for determining thermal expansion.

* Presented at the Fall Meeting of the Ohio Section, the American Physical Society, Mendenhall Laboratory, November 14, 1942.

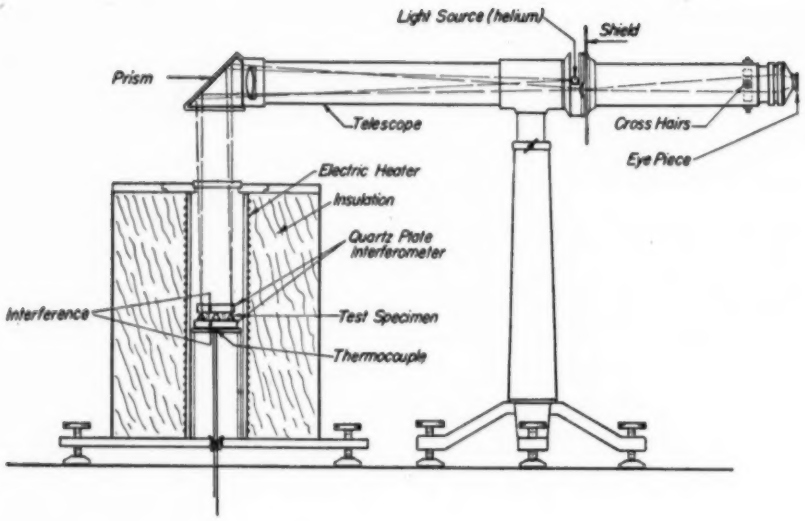

FIG. 2. Diagram of interferometer for determining thermal expansion.

the quartz tube and contacting the specimen indirectly with another smaller quartz tube or rod resting on top of the test specimen. The fused quartz has a low straight-line expansion curve over the area of measurement and a correction is made for it in the calculation of the expansion coefficients of the test material.

Figure 2 shows a diagram of the interferometer. The test pieces for the interferometer consist of three tetrahedra or pyramids of the test material which are cut and ground to the same height (about $\frac{1}{4}$ inch) or within 0.001 inch of each other. These are set up in the furnace as shown between two quartz plates which comprise the interferometer. The light source in the viewing telescope is a helium-filled bulb which gives a monochromatic beam: When the furnace is heated with the test pieces and quartz plates in position interference fringes are observed passing across the field. These are counted as they pass a given reference point and this number with the rate of temperature increase (preferably 2.5°C per minute) provide the data for calculating the coefficients of expansion. The advantage of the interferometer over the dilatometer is the extreme accuracy of the former and the fact that small specimens can be used.

Some typical thermal expansion curves are shown in Figs. 3 and 4.

DETERMINATION OF GLAZE FIT

There are several devices for the quantitative measurement of glaze fit which give a direct indication of the relative difference in thermal expansion between the body and glaze. One of these consists of heating in an electric resistance furnace a two-pronged bar of the fired ceramic body which has been glazed on the outside only and measuring the spread or contraction at the ends of the prongs of this tuning-fork-shaped test piece by means of a cathetometer.

Another method of measuring glaze fit employed at the Station utilizes a ring of the fired body about $2\frac{1}{2}$ inches outside diameter, 2 inches inside diameter, and $\frac{3}{4}$ inch high, glazed on the outside face only. Reference marks are placed on one edge of the ring about $\frac{1}{4}$ inch apart. Their distance apart is accurately measured with a microscope fitted with a micrometer eyepiece. A saw cut through the ring between the reference marks is then made with a circular metal saw impregnated with diamond chips on its periphery. The distance between the reference marks is again measured. An increase or decrease in this distance indicates, respectively, whether the glaze was under tension or compression before the ring was cut. The amount of this change indicates the degree of tension or compression. It is customary in making this test by either

method to determine the inherent strain, if any, in the fired body by testing an unglazed specimen.

THE SONIC PROPERTIES OF STRUCTURAL CLAY TILE

Structural engineers desire to know the characteristics of their building units as they purchase these units and other building materials under rigid specification. Specifications and test methods are the cornerstone on which engineering service operates. Most of these test procedures are methods and techniques originally devised by the physicist.

Very little is known regarding the characteristics of ceramic building units. It is known, however, that these units fail if improperly fabricated. Failure may have its origin in the raw materials used, in the mechanical processing, the drying, or the firing of the units. The structural materials division of the U. S. Bureau of Standards has been working on the problem of devising a method whereby the failure of improperly fabricated units can be predicted. They have worked out a program consisting of a schedule for running rigidly controlled freezing and thawing tests and have written a tentative specification based on these findings. Some ceramic structural materials are correctly evaluated by the specification, while others are not. The specification does not tell either how or why any of the materials fail.

When ceramic units are exposed to freezing and thawing conditions, and in fact to any temperature change, there is doubtless an internal friction set up. This friction may be great enough in time to cause strains that will ultimately rupture the unit. It would be desirable to know the trend toward failure before it has actually occurred. During the early stages of the changes which will ultimately cause rupture it would be desirable to be able to measure a property which would record this fact and especially so if it would record it in quantitative terms.

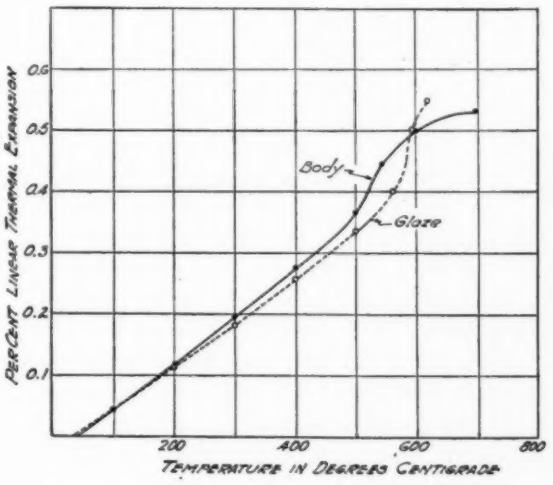

FIG. 3. Linear thermal expansion of dinnerware body and glaze.

A specific problem of this kind which has been undertaken at the Engineering Experiment Station of the Ohio State University is the prediction of failure of glazed brick and tile. Again, adaptations of research tools and techniques of the physicist are being employed in the determination of the sonic properties of these units.

It is a common practice among plant men to tap a product to determine whether it produces a good "ring." With certain products, such as grinding wheels, the sonic properties are being measured electrically to determine their fitness for service. In using the sonic method to evaluate the physical properties of an elastic body some constant or value which is indicative of the properties of the body material must be obtained. One which is usually obtained in sonic determinations is the modulus of elasticity on which the frequency of vibration depends. This may be readily determined for homogeneous simple laboratory test pieces such as straight rectangular bars which conform to the requirements of ratio of thickness to length (1 to 20 or more). However, this method cannot be used for shapes with a ratio of thickness to length which does not conform to the determined requirements.

The "damping properties of vibration" is an additional property of elastic bodies which is equally as important as the modulus of elasticity and may be determined by sonic methods on specimens regardless of size and shape. Excluding external friction the damping capacity is the amount of work dissipated into heat by internal friction in a unit volume of material during a cycle of vibration. A method of determining the specific damping capacity is by measuring the logarithmic decrement of vibration. This is the natural logarithm of any two successive amplitudes of a vibration whose amplitude decreases

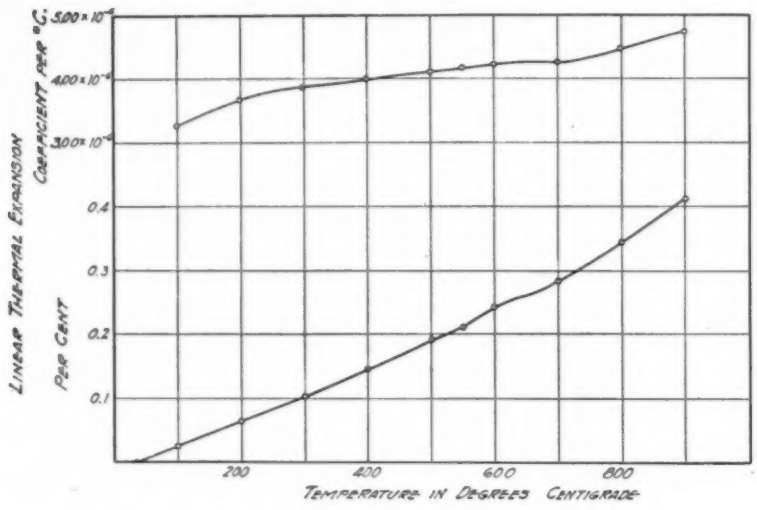

FIG. 4. Linear thermal expansion of a clay refractory.

exponentially and can be shown to be one-half the specific damping capacity.

In the investigation of glazed brick and tile it is planned to determine their damping capacity between every fifth cycle of freezing and thawing in the hope that the procedure will yield results from which curves can be drawn that will give information which will indicate the genesis and extent of failure of the unit.

The sonic measurement of the logarithmic decrement is a measurement of the resonance curve using the following formula:

$$e=\frac{\pi(f_2-f_1)}{f_0},$$

where e = log decrement, f_0 = maximum resonance frequency of vibration, f_2 and f_1 = frequency on either side of the maximum resonance frequency, at which points the amplitude of vibration is $1/\sqrt{2}$ times the amplitude at maximum resonance.

The term (f_2-f_1) is also known as ΔF and is the width of the resonance band at $1/\sqrt{2}$ of maximum amplitude.

The resonance method must be used in order to measure specimens of the size and shape of glazed structural tile and the log decrement is obtained as indicative of the specimens' properties.

Figure 5 shows a photograph of the complete apparatus used for this determination. This equipment includes a beat-frequency oscillator, suitable vibrator, microphone pick-up, cathode-ray oscillograph, and the necessary amplifiers.

The test specimen is supported on a foam rubber cushion so that it may be sustained in a free state despite the fact that its nodes of vibration cannot be calculated. Contact is made between the center of the specimen and the vibrator which is vibrated at the various controlled frequencies of the oscillator, and imposes this vibration in the specimen. The sound emitted by the vibrator and the vibrating specimen is picked up by the microphone and projected on the oscillograph screen. When the specimen is vibrated at its natural or resonant frequency a maximum amplitude is obtained. This point of maximum amplitude is shown on the screen and f_1 and f_2 are obtained by changing the frequency to obtain the respective amplitudes at $1/\sqrt{2}$ of the maximum. With these values the logarithmic decrement is calculated according to the formula.

THE SONIC PROPERTIES OF GLASS

Another study involving the determination of sonic properties has been made at the Engineering Experiment Station at the request of a glass company. In this case it was desired to learn the effect on decrement of tone of variations in the composition of the glass, and also the effect on

this property of annealing and disannealing the glass, and that produced by altering the dimensions.

It is customary to evaluate the quality of glassware such as a goblet by setting it in vibration and observing the tone produced. A goblet of good quality emits a tone which is pleasing to the ear, principally because the vibrations are sustained. If the vibrations die away rapidly the resultant tone is unmelodic and it is concluded that the glass is of inferior quality.

In the present investigation a method has been developed for making a quantitative measurement of the degree to which vibrations are sustained. This measurement was made in terms of the modulus of decay which is the time (in seconds) necessary for the amplitude or strength of vibration to decrease to 36.8 percent of its initial value (36.8 percent is the reciprocal of the base of the natural logarithms e, expressed in percent).

The four compositions of the glasses studied varied mainly in their lead and soda contents (Table I). The work was confined to test specimens in the shape of bars $\frac{1}{4}$ by 1 inch in cross section and of various lengths. The glasses were prepared, the test pieces pressed, annealed, and disannealed at the glass plant.

TABLE I. Chemical analyses and densities of glasses used.

	Type I	Type II	Type III	Type IV
SiO_2	73.4	69.2	63.0	56.0
R_2O_3	1.0	0.1	0.1	0.3
CaO	4.8	7.9	5.4	0.4
MgO	3.4	2.9	1.9	0.3
BaO	0.5			
Na_2O	14.2	13.5	8.0	
K_2O	2.7		5.7	14.6
PbO		6.3	15.7	28.5
Density	2.4601	2.6203	2.7795	3.0627

A correlation of tone quality and chemical composition was obtained by comparing the moduli of decay of bars of the same size but of different composition. To obtain some idea of how the shape of a goblet affects its tone quality the moduli of decay of bars of different lengths but of the same composition were compared.

The bar was set in vibration by a single blow delivered by a device actuated by a solenoid. The tone was picked up by a microphone and the pattern recorded on the screen of an oscillograph. A camera recorded the pattern.

The investigation used the following equation:

$$r = \frac{1}{k} = \frac{r}{f \log_e \frac{A_n}{A_{n+r}}},$$

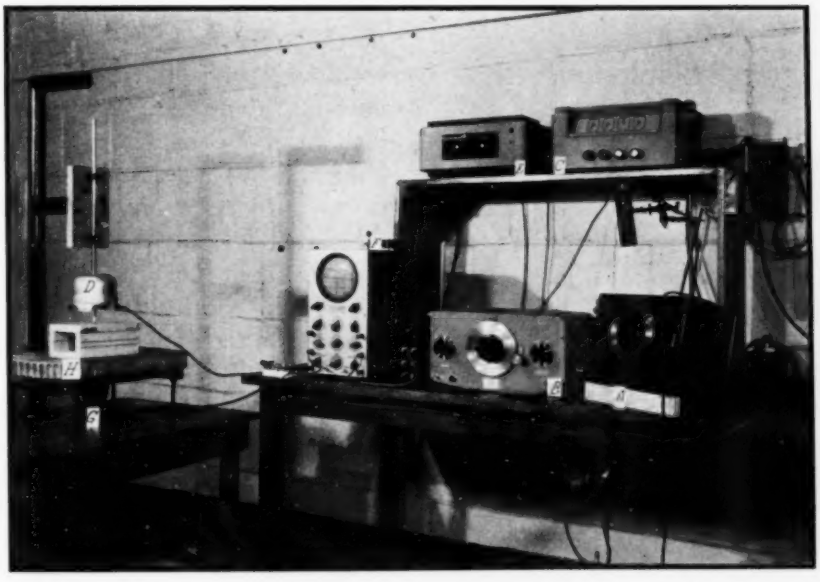

FIG. 5. Photograph of sonic equipment used for the determination of log decrement.

where r = modulus of decay, f = natural frequency of vibration of bar, A_n = amplitude of vibration at any time t, A_{n+r} = amplitude of vibration r cycles later.

This equation makes it possible to determine the modulus of decay by measuring the ratio of two amplitudes of vibration, the number of cycles elapsed between the two amplitudes, and the natural frequency of vibration of the bar.

The investigator on this study arrived at the following conclusions:

(1) Within the range of lengths tested, the modulus of decay is directly proportional to the length for bars of the same cross section and same chemical composition. This conclusion is significant because it indicates the possibility of improving the tone quality of goblets by altering their design.

(2) For bars of the same length and cross section but of different composition, the high lead content glasses have the best tone qualities.

(3) The slopes of the lines of the graph in Fig. 6 are such that for a given increase in length, the tone quality of the high lead content glasses is improved more than those of the low lead content.

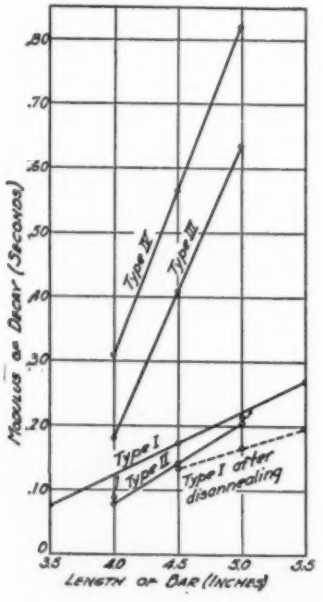

FIG. 6. Modulus of decay of glass samples.

(4) The effect of disannealing is to lower the pitch of the ring emitted by the vibrating bar and to decrease the length of time the bar continues to ring after it has been set in vibration.

THE ELECTRON MICROSCOPE

The electron microscope (shown diagrammatically in Fig. 7) is a new research tool which has its applications in all branches of the physical and biological sciences. It permits direct observations at magnifications far beyond the range of the optical microscope by focusing beams of electrons with magnetic lenses in much the same manner as is done with light beams in an optical microscope.

One application of this instrument is the study of surfaces by means of transparent replicas of these surfaces. Any depression in the surface is apparent as a dark area in a "negative" replica or as a light area in a "positive" replica of this surface, when viewed in this instrument.

This replica method in conjunction with the electron microscope is being tried in connection with a study of the scratch hardness of glazes. A scratch is produced by an instrument called a microcharacter. The test specimen on a movable stage is drawn under an accurately milled diamond point. The load on this diamond point is varied depending on the hardness of the glaze. The breadth of the scratch is measured under a microscope. With the optical microscope the magnifications of the width of a scratch produced by the microcharacter apparatus are insufficient to differentiate properly between glazes which have a moderate degree of resistance to scratching. Also, in the case of transparent glazes, erroneous measurements may be obtained because of the uncertainty of properly focusing on the surface, with the high degree of magnification required. These scratches are only 3 to 6 microns in width unless a sufficient weight is imposed on the diamond point to cause chipping of the glaze. The accuracy of the procedure depends on a track devoid of chipping. Using the electron microscope with magnifications up to 50,000 diameters by photography such scratches may be measured in inches instead of microns.

There are several ways of producing replicas of surfaces. One is to produce first a "negative" of the surface by condensing on the surface vaporized metallic silver evaporated from a

heating element in a vacuum chamber. This silver film is stripped from the surface mechanically and a "positive" replica produced by spreading a dilute solution of nitrocellulose over the surface, then removing the silver by dilute nitric acid. With this procedure difficulty is encountered in completely removing the silver without destroying the nitrocellulose film.

Another and much easier method of producing a negative replica which is used for observations directly is to produce the film on the original surface with a 0.5 percent solution of polyvinyl formal in dioxane. This forms a tough film which is easily stripped off the surface without deformation.

The only apparent objection to the use of the electron microscope for the measurement of scratch hardness is that a considerable number of replicas will need to be made of the surface along each scratch as each replica must be only about 2 mm in diameter when mounted for viewing in the microscope.

THE CYCLOTRON

Recently the problem of measuring the thickness of the gold band on dinnerware was presented for an immediate solution as the Federal government had specified that the band should not exceed a thickness of 1/10,000 of an inch. This thickness was of so small a magnitude that it could not be measured by any of the instruments available. A solution was therefore sought from the physicists and chemists. Because the time allowed for the solution of this problem did not permit of experimentation with unusual procedures, the technique of the chemist was employed, while recognizing that the procedure could not be expected to give an accurate result, under the circumstances. In this procedure, the gold band was dissolved off the ware and the gold deposited electrolytically. From the weight obtained the thickness of the band was calculated as a certain thickness, plus or minus 20 percent. This inaccuracy is readily explained by the facts that the metal deposit on the electrode was less than 2 milligrams, the gold lacquers used are not pure gold but a mixture of rare earth metals and the amount of these metals present is so small that the uncertainty of any results obtained is increased. In addition to these difficulties, the problems of voltages and current densities required for the depositions of the various rare earth metals were encountered.

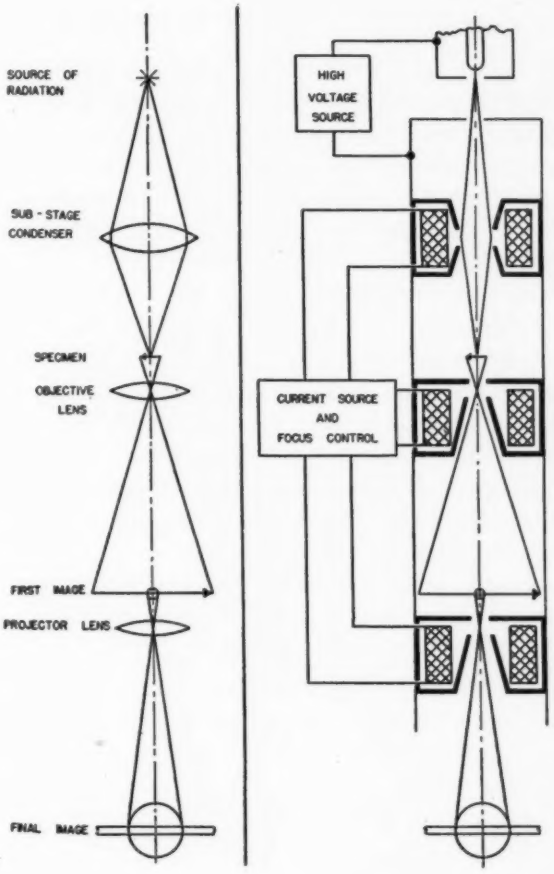

FIG. 7. Diagrammatic sketch of magnetic-type electron microscope compared with optical microscope placed upside down (left).

The cyclotron was suggested by M. L. Pool* as a possible solution of the problem. His proposed procedure was so simple and direct that it is worth recording here. He proposed to irradiate a section of the band with neutrons thus inducing an artificial radioactive gold. Simultaneously a piece of pure gold leaf would be bombarded with the neutrons. The radiation of the beta-particles from the band and the gold leaf would be counted in the Geiger counter.

* Associate Professor, Department of Physics, The Ohio State University.

What the masking effect of the other components of the system might have been was questioned at the time. It is regretted that the time allowed for this investigation did not permit investigating the use of the cyclotron.

ULTRASONIC VIBRATION AND CERAMICS

In the purification of whiteware clays and in some types of clay slip casting it is desirable to disperse the clay in water to its ultimate particle size, or at least to a finer particle size than is possible with electrolyte alone. The magnetostriction oscillator is an apparatus for producing ultrasonic vibrations which has been found, in laboratory trials, to produce this higher degree of dispersion in the case of the more easily dispersed whiteware clays. The magnetostriction oscillator utilizes the principle that nickel contracts when it is placed in a magnetic field and when it is subjected to a high frequency alternating magnetic field it produces ultrasonic vibrations. The small nickel tube of the apparatus is inserted in the suspension to be dispersed.

The experiments with clay slips showed that some of these, first dispersed with electrolyte, were further dispersed to a finer particle size when subjected to the ultrasonic vibrations. On the other hand, the vibration treatment alone without the electrolyte did not produce any finer particle size in most cases than did the electrolyte alone. A combination of the electrolyte and the vibration treatment produced a more dispersed system than did either one alone.

Another apparatus which produces higher frequency ultrasonic vibrations than the nickel tube apparatus utilizes a vibrating piezo crystal (quartz). A million cycles per second are possible with it. Its capacity is less, even, than the nickel tube apparatus but it should have a greater dispersing effect on a clay suspension. It is planned to try this newer technique on some clay suspensions.

Powdered Iron Cores

BY ALFRED CROSSLEY

Chicago, Illinois

THE importance of the use of powdered iron cores in electrical circuits is now being recognized, in view of the fact that there are available today cores made of many different ferrous materials suitable for use at frequencies from one to 150,000 kilocycles.

A brief historical review of the progress made in this new art is necessary in order that we may understand the reasons for the rapid advance made during the last decade.

The necessity for laminated steel in low frequency alternating-current machinery and transformers to obtain desired efficient operation was again demonstrated by Alexanderson in the United States and German engineers by the use of 0.001" thick silicon steel laminations in 10,000-cycle alternators and 30,000-cycle frequency multipliers. These devices performed a useful purpose in radio communications, but with the steady increase in the use of higher frequencies, the laminated steel was abandoned in favor of air core tuning devices until about 1920 when the Western Electric Company introduced iron dust cores (Toroids) for use in telephone filter circuits, said filter units having far greater efficiency than those which included the old type thin steel laminations.

These iron dust cores to the writer's knowledge were not used at frequencies in excess of 60,000 cycles for two reasons: First, the Telephone Company did not, in the main, require higher frequency services and, second, these early iron dust cores had particle size and insulation which would not provide efficient operation at higher frequencies (40,000 cycles and upwards). It was not until about 1927 that W. J. Polydoroff introduced powdered iron cores made from hydrogen reduced iron which provided efficient operation in radio tuning devices at frequencies as high as 1500 kc.

In 1930 Hans Voigt in Germany introduced powdered iron tuning inductances, which consisted of powdered iron particles sprayed on a sheet of paraffined paper which was rolled into a cylinder forming the core of a radio inductance. Later the German I. G. Farben Industries introduced a new type of powdered iron referred to as iron carbonyl, which due to its fineness and low losses is still being used in tuning devices operating at frequencies as high as 40,000 kc.

The hydrogen reduced and carbonyl iron powder have during the last nine years been used in the millions of radio receivers providing increased efficiency and compactness of design unheard of in any air core inductances.

Having briefly outlined the stages of advancement in the art, we should now consider the details of construction of powdered iron cores, mentioning first the various grades of powdered iron used in present-day tuning or filter devices.

In the low frequency operating region hydrogen reduced iron or permalloy materials are used. The hydrogen iron is produced by blowing iron oxide through a hydrogen furnace, which provides a pure iron particle having a diameter of from 1 to 40 microns. This iron is jagged in form, and, when compressed, remains in the shape it was compressed. Permalloy is a nickel iron, or molybdenum nickel iron alloy, which is mechanically reduced to required particle or grain size. It has very high permeability at low magnetizing field density, but loses this feature as the magnetic field strength is increased. Permalloy when properly processed into a toroid form has an effective permeability of 125, whereas under the best conditions the hydrogen reduced iron toroid will provide a permeability as high as 50.

In that part of the spectrum from 300 to 40,000 kc carbonyl iron will produce excellent results if the proper grade is used. There are three grades of carbonyl namely *C*, *E*, and *TH*. *C* grade (4 to 10 microns diameter) is most efficient in the lower part of the spectrum, *E* (2 to 7

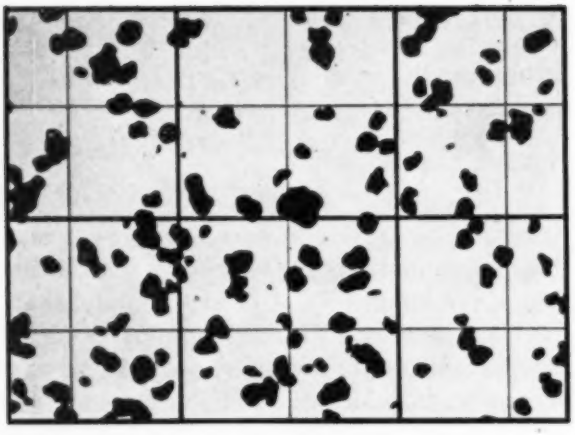

FIG. 1. Hydrogen reduced iron powder. Magnification 400×.

microns) the middle part, and *TH* (1 to 5 microns) the higher frequency section.

Carbonyl iron powder is obtained by decomposing liquid iron carbonyl into the metal particle, which is spherical in shape and of shell-like structure. Figures 1 and 2 show photomicrographs of hydrogen reduced and carbonyl iron, respectively, whereas Fig. 3 indicates the shell structure of carbonyl iron particles. This shell structure is one factor responsible for low eddy current losses in carbonyl powder.

A new type of iron has just been developed which is of an alloy base, having the same properties of carbonyl *TH*. This iron is referred to as IRN-16 and is most efficient in the band between 5000 and 40,000 kc. Another type of synthetic iron powder has been produced, which has an apparent permeability of 2 when used with the best form of cylindrical winding. This iron is referred to as IRN-8 and is used in the range 30,000 to 150,000 kc.

Magnetite (lodestone) a natural mined iron ore when ground to required grain size, for given frequency range, can be substituted for hydrogen and carbonyl iron powder, but due to its lower permeability is about 75 to 90 percent the efficiency of the metal powders.

In the manufacture of powdered iron cores the following procedures are necessary:

1. Selection of proper iron powder for required frequency range.
2. Insulation of each iron particle from each other.
3. The insulated powder is mixed with a binder and inserted into molds where it is compressed to desired form.
4. After molding, the cores are heat treated to complete polymerization or aging.

Each iron core manufacturer employs his own special insulation and means for processing the core and considers the technique a trade secret. This is only natural because there is no literature on this subject and they have learned a particular method of processing by the hard way, namely, by cut and try methods, which has taken many years to perfect.

Those who require iron cores for their particular application should specify the following to the iron core manufacturer so that they may obtain the most efficient core.

1. Frequency range of operation.
2. Size of inductance in millihenries.
3. Q or efficiency of inductance.
4. Size and type of metal container into which the inductance will be placed.

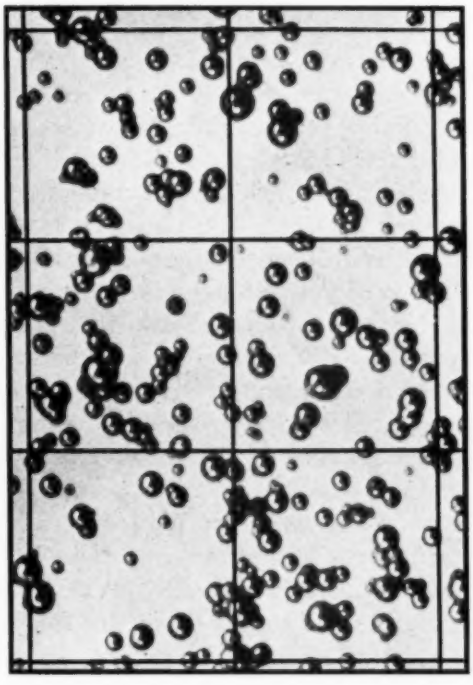

FIG. 2. Carbonyl iron "*E*." Magnification 500×.

Iron cores are made in various sizes and shapes, which generally are grouped in three classes, namely: cylindrical, toroid, and pot forms. Figures 4, 5, and 6 show, respectively, these general forms whereas Figs. 7, 8, 9, and 10 are the types with the winding form, which are commonly used in high frequency tuning circuits.

Figure 7 is the most common form and is found in millions of radio receivers, while the more efficient multiple coil unit shown in Fig. 8 is found in the more expensive receivers. Figure 9 is the standard cylindrical type, the core of which moves in and out of the coil and, in so doing, tunes the radio circuit to the desired frequency when a fixed condenser is employed. This latter method is referred to as permeability tuning and is rapidly superseding the more expensive variable condenser type of tuning. Figure 10 is the pot type wherein the coil is completely enclosed. The latter type when used with proper type of litzendraht cable is the most efficient type of high frequency inductance for use at frequencies higher than 50 kc. The writer has constructed a 200-microhenry inductance of this type which has a Q of 420 at 1000 kc.

Space does not permit a complete discussion of iron core tuning devices but it is permissible to mention a few outstanding applications where powdered iron cores produce remarkable results.

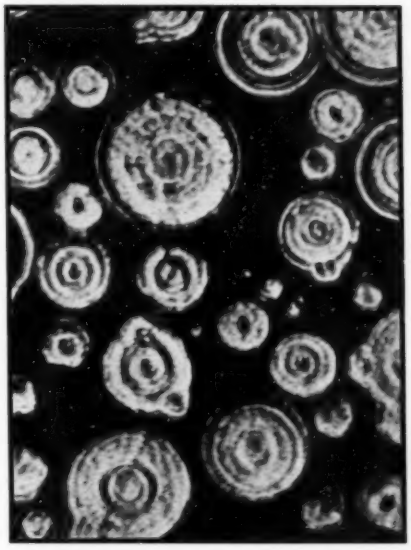

FIG. 3. Carbonyl iron "*E*." No. 43-s. Magnification 2000×.

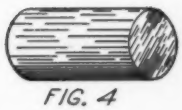

FIG. 4

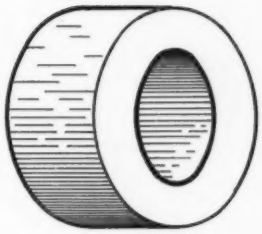

FIG. 5

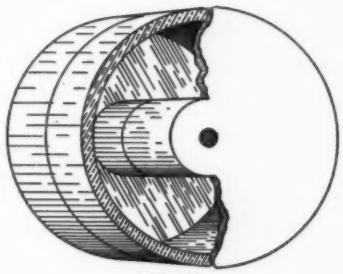

FIG. 6

FIGS. 4–6.

The first device is the cylindrical aircraft direction finding loop antenna (Polydoroff patent No. 2,266,262) using powdered iron cores. With this construction the same pick-up voltage is obtained with a $2\frac{1}{2}''$ diameter iron core loop as was possible with a 9″ air core loop. The smaller iron core loop reduces considerably wind resistance or drag over that of the larger air core type and in addition provides a better minimum or zero signal for the right angle position of the plane of the loop to the advancing wave front.

In this type of loop antenna the increase in efficiency is due principally to the increase in permeability of the media caused by the presence of powdered iron in the field of the coil. The reduction in size also cuts down the so-called "antenna effect" thereby producing a better minimum or zero (right angle) signal.

It is only natural to assume that all future aircraft loop antennas (direction finding or radio compass) will be of the iron core type for the obvious reasons previously mentioned.

The second application is a new device which will revolutionize audio- and radiofrequency

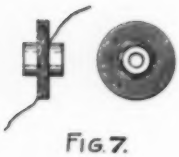

FIG. 7.

FIG 8.

FIG. 9.

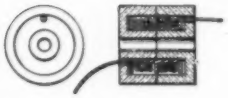

FIG 10.

FIGS. 7–10.

coupling systems. This device provides means for varying coupling from zero to a desired maximum value and in addition the phase relationship of said inductive coupling can be adjusted to be aiding or opposing the electrostatic coupling inherent in the system being coupled.

Figure 11A is a schematic drawing of the conventional coupling transformer used in the radio receivers, comprising a primary winding with its tuning capacitance and a secondary winding also with the tuning capacitance both of which permit transfer of power from the plate circuit of one vacuum tube to the grid circuit of the succeeding tube. It is common practice to couple magnetically one coil to another by moving one coil to a given distance from the other so as to obtain critical coupling. Increasing the coupling beyond critical coupling, introduces complications, because the mutual inductance starts increasing both primary and secondary inductance and both circuits are thereby tuned to a lower frequency.

Figure 11B shows a means for compensating for increase in mutual inductance by decreasing automatically the inductance of the primary and secondary when the mutual inductance or coupling is increased or vice versa. In this circuit three powdered iron cores linked together are associated with the following respective parts of the circuit: first, a small coil in series with the primary; second, a coupling coil; and third, another small coil in the secondary circuit. The movement of the iron core into the field of the coupling coil also produces a movement out of the field of both primary and secondary coils providing an increase in coupling or mutual inductance, but at the same time a decrease in inductance of the primary and secondary inductances thus keeping constant the total inductance of both primary and secondary circuits.

The maintenance of a constant inductance in both circuits with variation in coupling between circuits maintains the tuning of both circuits to a fixed frequency and in addition provides means for obtaining extremely sharp tuning or any intermediate coupling up to the over-coupled condition necessary for high fidelity reception. With such a system the side band acceptance is uniform because there are no distorted side band characteristics. Figure 12 indicates the receptivity characteristics of the system for the varying degrees of coupling.

Figure 11C is an improvement electrically and economically over arrangement shown in Fig. 11B. In this figure one coupling coil system and one iron core do the work of three previously used. The coil consists of a Bakelite tube upon which is wound bifilarly four coils and inside of the tube is a powdered iron core having an adjusting rod. The winding connections of one of the secondary coils are reversed thereby providing the novel condition whereby the movement of the iron core can produce any desired

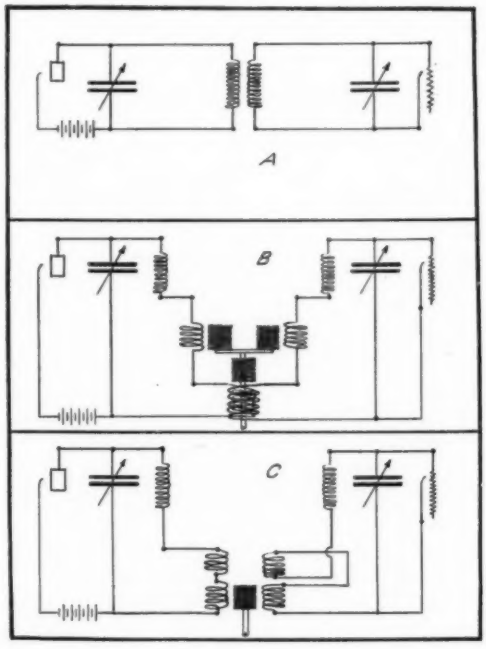

FIG. 11.

degree of coupling from zero to maximum with the additional feature of phase changing.

Zero coupling is obtained by moving the core to the center of the tube whereby the negative mutual obtained with the reversed primary coil is counterbalanced by the positive mutual or coupling of the lower coil system. Moving the core upwards produces an increase in coupling until maximum is obtained when the core is completely enclosed by the upper coil system. Moving the core downwards from the center position also increases the coupling but this coupling is 180 degrees out of phase to the previously mentioned coupling.

Throughout the entire movement of the core for any degree of coupling the inductance of both the primary and secondary remains constant.

This coupling system can be applied to oscillating systems for positive control of feedback. It also can be used with amplifier and filter circuits where negative or positive control of coupling or feedback is required.

In concluding it is permissible to say that we have, in the powdered iron core, a tool which permits us to obtain three important conditions: first, increase of inductance with increased efficiency; second, tuning of or coupling changes in circuits by movement of iron core; and third, increase in signal voltage in loop antennas by increase in permeability of the media. These functions, when properly applied, will open up new fields for research in both audio- and radio-frequency circuits.

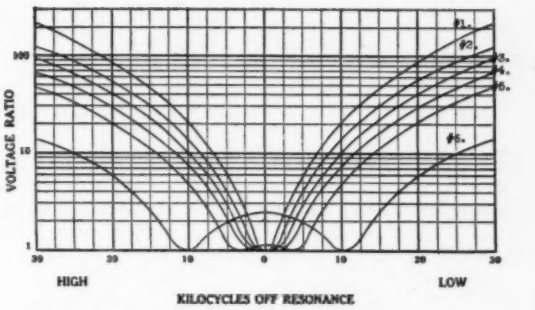

FIG. 12. Variable coupling interstage transformer. Frequency 456 kc. 1.5 millihenries, primary and secondary. No. 1, under-coupled, gain 125; No. 2, under-coupled, gain 263; No. 3, optimum coupled, gain 286; No. 4, over-coupled, gain 286; No. 5, over-coupled, gain 250; No. 6, over-coupled, gain 112.

Calendar of Meetings

September

6–10 American Chemical Society, Pittsburgh, Pennsylvania
12–13 American Mathematical Society, New Brunswick, New Jersey
21–24 Technical Association of Pulp and Paper Industry, Chicago, Illinois

October

7– 9 Optical Society of America, Pittsburgh, Pennsylvania
18–23 American Welding Society, Chicago, Illinois
23 American Physical Society, Metropolitan Section, New York, New York
29–30 Society of Rheology, New York, New York

November

14–16 American Institute of Chemical Engineers, Pittsburgh, Pennsylvania
26–27 American Physical Society, Chicago, Illinois
26–27 Central Association of Science and Mathematics Teachers, Chicago, Illinois

December

27–Jan 1. American Association for the Advancement of Science, Cleveland, Ohio

Frictional Phenomena. XV

BY ANDREW GEMANT
The Detroit Edison Company, Detroit, Michigan

Chapter XV. External Friction of Solids

Abstract

IN this chapter external friction between sliding solids is discussed. The fundamental mechanism, which is based on the geometric irregularity of macroscopically plane surfaces, is discussed first. A review of the experimental facts on the dependence of the friction coefficient on pressure, speed, and temperature reveals the plastic deformation of the surface irregularities as a complicating factor during sliding. Further experimental material on the so-called slip-stick motion, and on observations of traces of soft metals left upon harder metals sliding against them indicate abrasion as another complicating factor in external friction.

57. Basic Mechanism of External Friction

The last field to be dealt with in this monograph is the friction experienced when two solid bodies are shifted against each other. This type is certainly the first that came to any observers' attention as it is the most obvious and most frequent in both everyday and scientific processes. Other fields, such as liquid viscosity, were conceived and explored at a later time, and were distinguished as internal friction, in contradistinction to the external friction that occurs at the boundary between two solids in relative motion.

In spite of this primary nature of external friction it has been placed at the end of this treatise, the sequence in which, as has already been stated, is one of increasing complexity. And indeed, external friction, which was observed first, is apparently the most complicated as to its mechanism. While it has been possible to present a fairly satisfactory, although often far from complete, theory of gaseous, liquid, non-Newtonian, plastic, and internal solid friction, this is by no means possible in the field of external solid friction. There simply does not exist any generally accepted theory; all we have are only first and very fragmentary attempts along this line.

The procedure to be followed in this chapter will be this: First, a very elementary mechanism underlying external friction will be presented. This mechanism is most likely correct, although really it is only a first step to a theory since probably a large number of complicating processes play a more or less important role in individual cases. Next, the existing experimental material, (which, too, by the way, is very incomplete, and at the best only semi-quantitative) will be presented. In discussing the experimental facts, the various additional complications in the mechanism of friction will be considered in such a manner as to give an explanation for the various experimental observations. This procedure is about the best that can be followed with our present knowledge.

The basic mechanism of external friction originates from Coulomb. It is known that all solid surfaces are rough, the roughness being either macroscopic (dimension of asperities or protuberances $\sim 10^{-2}$ cm), or microscopic ($\sim 10^{-3}$ to 10^{-4} cm), or even submicroscopic ($\sim 10^{-5}$ cm). The dimensions of the asperities along a surface are probably randomly distributed over certain average values, but for sake of simplicity we shall operate in the following with these average dimensions. Two horizontal surfaces, A and B, sliding on each other are shown schematically in Fig. 142, A moving to the right, and B being at

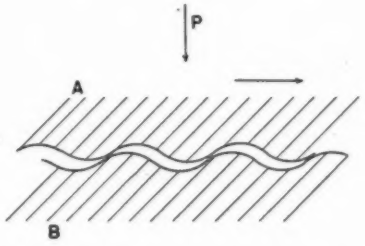

FIG. 142. Schematic diagram of two bodies sliding upon each other.

rest. The weight of A, shown by the arrow P, presses A against B with a pressure $p=P/a$, if a is the contact area.

If A is moved with a constant speed to the right, the hills of the top surface will drop periodically into the valleys of the lower surface. If the average distance of the asperities is λ cm, then the frequency of this periodic motion will be $1/\lambda$ per cm of path covered. Because of the irregularity of the surface pattern, the phase of this periodic motion will be different for the various asperities, even if the period is approximately the same for all. Such a coexistence of different phases of motion among the asperities is obviously possible only if deformations, whether elastic or plastic, of the asperities take place. As a first rough approximation, however, these deformations will be disregarded.

To raise a hill out of a valley into which it has dropped, in order that horizontal motion may be maintained, will require a certain force applied horizontally. The magnitude of this force per surface unit F_0 can be estimated by considering a single hill of A with a base area λ^2. The horizontal force acting on it is $F_0\lambda^2$, the vertical pressure is $p\lambda^2$. Consider a motion of this hill as it is lifted from a valley of B to a hill-top. Such a motion will in reality not require a constant force, since the latter depends on the slope of the hill which varies from point to point. If, however, $F_0\lambda^2$ represents the average of the force required to move a hill from one valley to the next, its work is given by $F_0\lambda^3$, since λ is the total path for each period of the motion. The potential energy obtained is $p\lambda^2 h$, if h is the height of the hill. Hence, equating the two, we have

$$F_0=(h/\lambda)p. \qquad (166)$$

Here each side of the equation has been divided by λ^2, in other words, the relation, valid for a single hill, has been averaged over the surface unit.

The question immediately arises: What happens with the gained energy during the downward motion that follows each lifting? When a hill of A passes over the top of a hill of surface B, it will be pushed down into the next valley owing to the pressure p. This impact is now assumed to be essentially inelastic, i.e., the potential energy $p\lambda^2 h$ will be converted into heat. A mechanism of this kind, in which the work done by the external force is dissipated as heat, must lead to the observation of a friction force, the magnitude of which is given by Eq. (166).

It is quite obvious that this crude picture of external friction based solely upon geometric consideration of the sliding surfaces, can be considered as a framework at best. The asperities are certainly not rigid, as assumed here, but, as already mentioned, will undergo elastic deformations, both in the direction of and normal to the surface. Several other complicating factors will become evident from the next section. The conception of external friction as a superposition of a number of factors upon a basic mechanism was also emphasized recently by Claypoole and Cook.[1]

58. *Influence of Experimental Factors on External Friction*

(1) *Effect of Pressure and Area of Contact*

One of the fundamental laws of external friction, first stated by Amontons, relates the total frictional force F, acting between two bodies, to the weight P of the top body or, more generally, to the total pressure, acting normal to the boundary. The relation states that the two quantities are proportional and is expressed by

$$F=\mu P. \qquad (167)$$

The proportionality factor is called the coefficient of external friction, and it is a quantity depending only upon the nature of the two surfaces in question.

This figure has not great significance as to its absolute value, since it depends on many factors,

TABLE XIX. Coefficient of external friction μ for various materials, after Tichvinsky, and Schnurmann.

Combination of materials	μ
Carbon —Glass	0.18
Copper —Mild steel	0.36
Garnet —Mild steel	0.38
Glass —Glass	0.40
Hard steel—Hard steel	0.42
Cadmium —Mild steel	0.46
Ebonite —Glass	0.53
Mild steel—Mild steel	0.57
Copper —Copper	0.60
Nickel —Mild steel	0.66
Cadmium —Cadmium	0.80
Aluminum—Aluminum	1.4

but for sake of illustration a few data, as measured by Tichvinsky[2] and by Schnurmann,[3] for dry surfaces are given in Table XIX. It can be seen that the figures are of the order 0.2–0.7, but figures higher than 1.0 are occasionally observed. Figure 143, also after Tichvinsky, shows

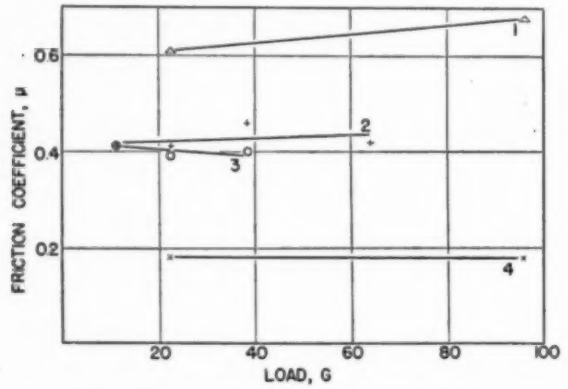

FIG. 143. Friction coefficient μ *vs.* load P, for various materials: 1. Nickel—mild steel; 2. Hard steel—hard steel; 3. Glass—glass; 4. Carbon—glass. (After Tichvinsky.)

that the coefficient is, indeed, independent of the load, proving that Eq. (167) is correct.

Amontons' law contains the additional statement that the total frictional force F for a given total load is independent of area of contact, a statement often verified experimentally.

Both these statements follow directly from Eq. (166) of the previous section. Multiplying both sides of this latter by the area of contact a, Eq. (167) immediately obtains, with

$$\mu = h/\lambda. \tag{168}$$

Thus μ, the friction coefficient, depends only upon the nature of the surface, as has been found experimentally. The relation (168) is, of course, just as crude as was Eq. (166).

In practical cases the surface of contact between two bodies is often curved. The actual area of contact in such cases then increases with the total weight P. Yet, Eq. (167) is valid: The frictional force is proportional to the total pressure and is not affected by the variation of the contact area with pressure.

(2) *Velocity of Sliding*

Observations show that the coefficient of friction for low relative speeds is approximately independent of the speed. This rule was verified by Tichvinsky, among others. He took deceleration curves which, when analyzed, give directly the friction as a function of speed. If the two are independent, deceleration is constant. He found the latter to be true between relative speeds of 60 and 600 cm per sec.

The rule appears to break down for higher speeds, in that μ decreases with increasing velocity. The speed limit at which the constancy of μ ceases appears to vary considerably with the materials investigated. Some data are shown in Fig. 144, as compiled by Stanton.[4] The coefficient is shown as a function of relative speed in mph (1 mph = 44.7 cm/sec.). Curve 1 refers to cast iron brake blocks on steel wagon wheels, curve 2 to locomotive driving wheels on steel rails in foggy weather. (This last qualification is essential, since water vapor introduces boundary lubrication; see Section 59 (2), also Chapter X.) It should be added that curve 2 refers to so-called rolling friction; although important from a practical standpoint, this type will not be dealt with in this monograph.

It is customary to differentiate between static and dynamic friction coefficient, the first corresponding to the force just necessary to overcome friction and start sliding, the second to the force necessary to maintain motion. It appears that under ideal conditions (perfectly dry, hard surfaces) the static coefficient coincides with the

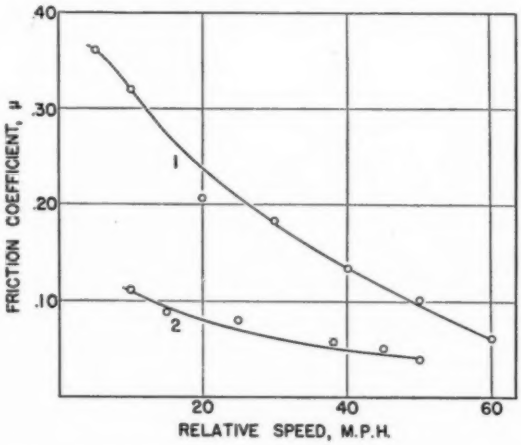

FIG. 144. Friction coefficient μ *vs.* relative speed: 1. Cast iron brake blocks on steel wheels; 2. Steel wheels on steel rails in foggy weather. (After Stanton.)

dynamic coefficient for low speeds. In most practical cases, however, an abrupt drop in the value of μ takes place, as soon as motion starts. This effect is probably a secondary one, due to abrasion of the tips of the hills.

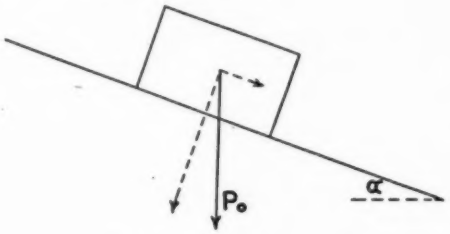

FIG. 145. Angle of repose.

If the force acting upon a body that is detained by external friction increases from zero, no motion occurs until the force corresponding to the static coefficient is reached. This force may be looked upon as a yield value, which is very well defined in this case, in contradistinction to plastic motion where it was shown to depend on the sensitivity of the measuring device.

The static coefficient is related to the so-called angle of repose, which is obtained by tilting the two bodies until sliding sets in (Fig. 145). Let us denote this angle by α. With P_0 = weight of the top body, the total pressure acting upon the lower surface is $P_0 \cos \alpha$ and the total driving force is $P_0 \sin \alpha$. Substituting $P_0 \cos \alpha$ for P and $P_0 \sin \alpha$ for F in Eq. (167), we obtain:

$$\mu = \tan \alpha. \tag{169}$$

The angle α is called angle of repose; by means of Eq. (169) it allows the static value of μ to be determined.

In the light of the discussion above it is evident that accurate figures on μ should be specified whether the static or a kinetic value is meant and, in the latter case, at what relative speed.

The experimental fact that the frictional force is initially independent of the speed and decreases at high values of the latter is in qualitative agreement with the Coulomb mechanism. It can be seen that Eq. (166) does not show the speed explicitly. However, it can be shown that for high speeds a decrease in the effective value of h, and thus of μ, might be expected. For small speeds the asperities will drop to the bottom of the valleys, whereas for high speeds the time will not be sufficient for a drop of this extent to take place and the asperities will hit instead the ascending branch of the next hill. Thus the effective depth of the valleys will be lowered. While this explanation seems to be correct qualitatively, it does not hold quantitatively. A decrease of μ can be expected when the order of speed becomes

$$u_0 \sim \lambda (g/h)^{\frac{1}{2}} \tag{170}$$

(g = gravitational constant). With λ and h of the order 10^{-3} cm the value of u_0 would be of the order of 1 cm/sec., whereas the experimental data in certain cases indicate a higher order than that. This also shows that other processes apart from a mere lifting against the normal pressure take place during sliding; these will be discussed below.

It should be added that there are exceptions to the above-mentioned typical variation of μ with speed, particularly in the case of rubber. While the friction coefficient at speeds of less than 10^{-3} cm/sec. is less than the static coefficient, for higher speeds the coefficient increases with the speed. This behavior is brought out clearly in a recent paper by Roth *et al.*[5] This exceptional behavior of rubber is not surprising in view of its low elastic modulus, making the simple concept of lifting of asperities of minor significance in this case.

Interesting results are obtained if the dependence on speed is considered in connection with the analogous effects in other frictional processes previously discussed in this monograph.

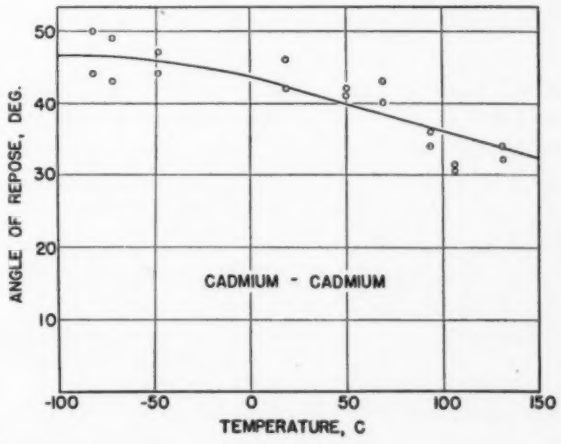

FIG. 146. Angle of repose *vs.* temperature for cadmium—cadmium interface, according to Schnurmann.

However, it will be more useful to discuss this matter after the temperature effect has been considered.

(3) *Effect of Temperature*

Although the question of the temperature influence is of considerable practical and theoretical importance, the information upon this point is very scanty. Schnurmann[3] recently investigated this question by using materials that were baked out at 160°C and then vacuum-treated in order to remove absorbed layers of vapors. Some of his data are shown in Fig. 146. They refer to a cadmium-cadmium interface, and show the angle of repose α as a function of temperature. While the angle (and μ) appear to be fairly constant below room temperature, above this up to 150°C a slow decrease takes place, the value of $d\alpha/dt$ being -0.08. It should be recalled that cadmium has the low melting point of 321°C; thus the range of decreasing μ is not too far from the melting point. Other metals of higher melting point may not show a decrease near room remperature.

Merchant[6] investigated polished pairs of like metals; when they were tested below the annealing range, μ was found to be independent of temperature.

The simple model explained above does not indicate any influence of temperature on external friction. Thus, it appears that the model might approach reality at lower temperatures, but has to be modified at higher ones. The modification consists in introducing the plasticity of both or at least one of the sliding bodies as a factor. The asperities do not have a constant shape, but may be plastically deformed under the action of the tangential force. The chief result of this mechanism is a reduction of the effective value of h, the height of the asperities and, hence, of the friction coefficient μ. An accurate evaluation of this influence of plasticity on external friction has not yet been carried out. It will become effective if at least one of the materials has a finite plasticity—for instance, if it is nearing its melting point. The effect obviously increases with increasing temperature (decreasing plastic resistance) in agreement with the behavior shown in Fig. 146.

At this point of the discussion it will be useful to review briefly the different friction mechanisms encountered in this monograph. Table XX gives a summary. The first column shows the state of the materials. The last item in that column contains the specification "dry" in order to differentiate it from lubricated surfaces (see Sect. 59 (2)). The second column shows the type of mechanism underlying friction, the third the relation between stress and rate of shear, and the fourth the relation between frictional force and temperature.

The chief point to be emphasized is that as long as viscosity is controlled by the kinetic energy of molecules, the stress increases with rate

TABLE XX. Summary of various frictional processes.

State of material	Mechanism of friction	Stress *vs.* rate of shear	Frictional force *vs.* temperature
Gaseous	Molecular, purely kinetic	Increasing, linearly	Increasing, according to $\frac{1}{2}$ power
Liquid	Molecular, potential barriers *vs.* kinetic energy	Increasing, linearly	Decreasing, exponentially
Liquid, non-Newtonian	Molecular + submicroscopic, potential barriers *vs.* kinetic energy	Increasing, less than linearly	Decreasing, exponentially
Solid, plastic	Molecular + submicroscopic, potential barriers *vs.* kinetic energy	Increasing, logarithmically	Decreasing, exponentially
Solid, surface dry	Submicroscopic + microscopic, potential barriers	Independent, secondarily decreasing	Independent, secondarily decreasing

of shear and, obviously, there will be a pronounced variation of viscosity with temperature. An explanation for the first of the above two statements can be formulated generally in the following way. Kinetic energy involves the validity of statistical laws, hence the number of moving molecules (rate of shear) will vary according to the Boltzmann function, the potential energy in which is both internal and external (stress). When kinetic molecular energies cease to play a role, as in external solid friction, there will be no variation of stress with rate of shear (unless secondarily), and also no variation of the stress with temperature (unless secondarily). The process is then controlled by potential barriers alone and the rate of shear does not enter into the picture. These interesting relations are best illustrated by the plastic resistance of solids, which occupies a transitory position between frictional forces controlled by kinetic and potential energies, respectively. It was shown in Section 42 how the plastic stress approaches a constant value as the elastic potential energy becomes commensurable with the potential energy of the barriers; in this stage the kinetic energy has lost its significance.

The trend shown by the stress *vs.* rate of shear relation for various states of matter is shown

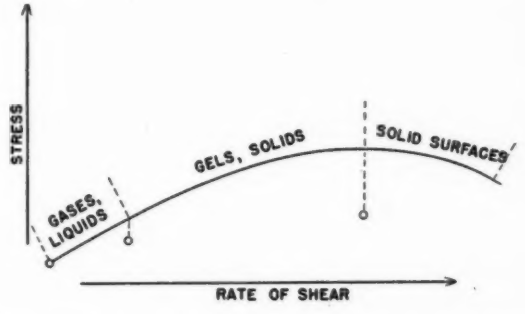

FIG. 147. Schematic diagram of stress *vs.* rate of shear, for various states of matter.

diagrammatically in Fig. 147. The coordinate system has to be shifted along the curve in order to show the behavior of the various states of matter, indicated along the curve. In order to include yield values, in particular the static friction coefficient, the origin of the coordinate system has to be placed at the three respective circles shown in the diagram.

59. *Further Experimental Observations on External Friction*

(1) *The Springiness of the Sliding Parts*

Interesting effects are observed if both parts are not rigidly mounted, one of them possessing a certain flexibility. In order to fix our ideas let

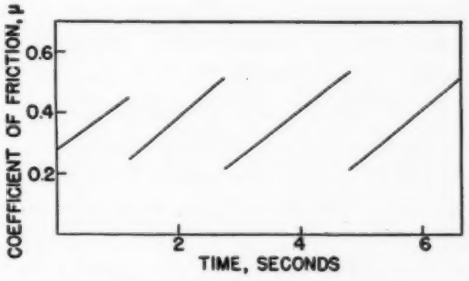

FIG. 148. Coefficient of friction *vs.* time in a stick-slip process for bearing metal on steel. Motion from left to right. After photogram obtained by Bowden *et al.*

us assume that the lower surface is rigidly attached to a mechanism (a piston, for instance) and has a constant speed. The upper surface, which is at rest, may be mounted rigidly or not. In the former case the relative speed is strictly constant at any time. In the second, when the upper body is mounted by means of springs, a discontinuity of the motion is observed. These are the results of certain vibrations exhibited by the top body. Such vibrations, which are caused by a steady motion controlled by external friction, have been known for a long time and have practical significance. This latter aspect will be discussed in Chapter XVI, while some basic experiments, with their results, will be presented in this section. These were carried out by Bowden *et al.*[7,8] and more recently by Muskat and collaborators.[9]

If the motion of the top body is registered by means of a moving film camera, pictures like that shown in Fig. 148 are obtained. This figure was drawn after a photogram obtained by Bowden. The abscissae are units of time (seconds) and the ordinates are displacements that have been converted into force, and, by dividing by the load, into friction coefficient.

The top surface is first dragged along with the lower one. This can be seen from the slope of the line, which, in terms of displacement, is found

to be identical with the speed of the lower surface. The elastic restoring force of the top body then increases, obviously until the static friction force is reached (in the figure at about 0.5). What happens next depends upon the friction force *vs.* velocity characteristic.

If the force were independent of, or were to increase with, the velocity (represented by a so-called positive characteristic), then a stable equilibrium would be attained. In case of independence, a certain deflection of the top spring would result independently of the relative speed; in case of a positive characteristic, the deflection would increase with the speed, but the equilibrium would be stable and the motion stationary in both cases.

If, however, the friction force decreases with increasing velocity, as represented by a so-called negative characteristic (see the third section of the curve in Fig. 147), then no stationary equilibrium is possible. As a relative speed develops, the friction force drops, resulting in an acceleration of the top body toward its normal position. Owing to this impulse the latter will perform vibrations in its natural frequency, which will decay according to the total damping present. The natural frequency was high in Bowden's case; thus these vibrations are not visible in Fig. 148. In one figure published by Bowden, however, these oscillations were clearly visible. When the relative speed again drops to a low value, the friction force increases and the top body again is dragged along, as seen from the figure. The whole process is repeated again. Hence, as has been explained, this remarkable periodicity, termed stick-slip, is caused by the negative characteristic of force *vs.* velocity.

It should be added that if the vibrational velocity of the elastically supported body and the velocity of the continuously driven body become of the same order of magnitude, the resulting stick-slip process will develop into undamped, self-excited vibrations, corresponding to the natural frequency of the elastic body (see Section 63).

Useful information is obtained from measurements of the contact resistance between the two bodies. They indicate that during the stick-phase, as the force of friction increases, the points of contact undergo some changes. This is brought out clearly on the two diagrams shown in Fig. 149, drawn after photographs of Bowden. They

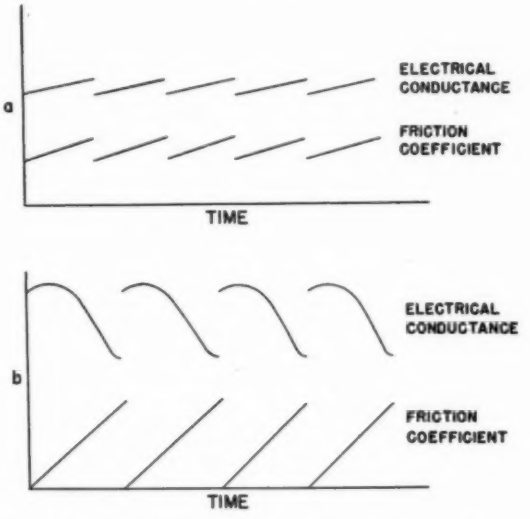

FIG. 149. Coefficient of friction and electrical conductance *vs.* time in a stick-slip process. After photograms obtained by Bowden *et al.* (a) Constantan—steel. (b) Tin—steel.

show simultaneous variations with time of the friction coefficient and the electrical conductance of the contact. Diagram (a) refers to constantan-steel, (b) to tin-steel. Increasing conductance points to increasing contact area of the protuberances. A change in the contact area probably results from a deformation of the points of contact.

In the first of the two cases shown, increasing frictional force flattens out the points of contact and conductance increases. When the slip takes place, the conductance drops rapidly to the original value.

In the second case the drop in conductance occurs toward the end of the stick-phase. Bowden explains this by assuming that welding of the contact points takes place, and that these minute welds become extended toward the end of the stick-period, thus causing a drop in the conductance.

The question of welding is a controversial point in the theory of friction. It appears[10] that a raise in temperature to a sufficiently high level for a welding to take place is quite exceptional. Such a temperature rise might happen if the sliding velocities were particularly high, or if the melting points of the components were particularly low. Because of the high thermal con-

ductivity of metals, it is not very likely that locally high temperatures could be obtained. Nevertheless, the possibility of this complication should be kept in mind, for instance, at very high pressures at which the sliding surfaces stick. This so-called seizure pressure depends upon the finish of the surfaces.[11] However, welding cannot be considered as a generally present feature in friction; it certainly does not take place during friction of non-metallic surfaces, like wood.

Even if the theory of a welding of contact points does not hold generally, there is a possibility of contact points of either of the two bodies approaching the other surface to such a degree that molecular attraction forces between the two bodies become active. It is known that such forces extend only to a distance of the order 10^{-7} cm, but an approach to this degree might take place during the stick-period. If this is admitted, then the simple Coulomb model, based on geometric considerations, has to be modified by taking into account the molecular attraction as part of the frictional force. According to this idea the molecular attraction forces have to be overcome each time a barrier is passed. Tomlinson[12] worked out a mathematical theory along these lines. It is not easy on the basis of available experimental material to decide whether such a mechanism plays an essential role or not. Certainly in those cases in which there is little or no dependence of friction on temperature this mechanism cannot be too important.

Whatever the role of local welding or molecular attraction forces in certain cases, their presence cannot be concluded from the mere fact of the stick-slip process. The negative friction *vs.* speed characteristic is, as pointed out above, sufficient to produce the effect.

(2) *The Nature of the Sliding Surfaces*

Further interesting observations are made if the sliding surfaces are investigated microscopically after sliding has taken place.

If the two materials have widely differing hardnesses (as in the case bearing metal and steel), then the softer material often exhibits characteristic parallel lines upon its surface in the direction of sliding. These lines can be interpreted as grooves cut by the harder material on the softer. Such lines are observed not only with two dissimilar metals, but also with glass-metal combinations, as shown by Schnurmann.[10] The harder of the two metals shows characteristic traces of the softer metal upon its surface after sliding.

These observations indicate the most radical departure from the simple geometric model. They clearly show that asperities, when lifted upon each other, do not always glide over the top, nor is the only possible complication a deformation but that a breaking away of the protuberances often also takes place. The work spent against the frictional force in such a case is used partly in the creation of new surfaces, this being to a certain extent an exception to the general rule that work against friction forces always appears as heat.

When a harder metal slides upon a softer one, it is chiefly the asperities of the latter that break off and then adhere to the harder; hence the grooves appearing upon the softer metal and the traces appearing on the harder. The final effect will be that the small amount of the softer metal that has been abraded now acts as a sort of lubricant between the two surfaces, a rather beneficial effect from the practical standpoint. The greater the difference in hardness between the metals the more pronounced will be the effect. Soft metals of this type, copper, tin, or lead alloys, are used as bearing metals.[13]

However, if the two surfaces are the same, each will abrade the other, no soft lubricant will be produced, and the wear of such a pair of equal metals will reach much higher grades than if one of the pair is softer. The practical implication of this conclusion is obvious.

It is not only the hardness of the sliding surfaces that affects the friction. Schnurmann[14] stresses the fact that an adsorbed layer of water or other impurity is often present upon solids and will considerably influence the friction coefficient. This effect[15] was discussed in some detail in Section 40 of Chapter X. While it deserves consideration from a practical standpoint, it is believed that the adsorbed layers are not generally characteristic for the mechanism occurring between unlubricated, dry surfaces, which for this reason is often called dry friction. This latter is

more adequately described by the three elementary processes taking place at the protuberances: lifting, deformation, and abrasion. Other complications, such as welding or boundary lubrication by absorbed layers, are rather incidental. In the latter group of complications belongs also a component of friction due to contact electrification, observed by Schnurmann and Warlow-Davies[16] on electrically insulated surfaces—even with a pair of metals.

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Errata: Flow and Recovery Properties of Viscose Rayon Yarn

[J. App. Phys. **14**, 224 (1943)]

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THE calculation reported on page 229 should be corrected to read as follows:

¾ Percent Extension

$$b = 2.35/4 = 0.59$$
$$0.35\text{g/d} = A - 0.59 \log (10^3+1)$$
$$A = 2.12$$
$$S = 2.12 - 0.59 \log (t+1)$$
$$S = 2.12 - 0.59 \log (10^{-2}+1)$$
$$S = 2.12\text{g/d}$$
$$\therefore E_{1\%}{}^* = \frac{2.12\times 1.33\times 10^9\dagger}{0.75}$$
$$= 3.8\times 10^{11} \text{ dynes/cm}^2.$$

4 Percent Extension

$$b = 2.25/4 = 0.56$$
$$0.55\text{g/d} = A - 0.56 \log (10^3+1)$$
$$A = 2.23$$
$$S = 2.23 - 0.56 \log (t+1)$$
$$S = 2.23 - 0.56(10^2+1)$$
$$S = 2.23\text{g/d}$$
$$\therefore E_{4\%}{}^* = \frac{2.23\times 1.33\times 10^9\dagger}{4\%}$$
$$= 7.4\times 10^{10} \text{ dynes/cm}^2.$$

* These values do not represent instantaneous moduli but rather an average of the composite moduli in effect during the respective extension.

† 1 gram/denier = 1.33×10^9 dynes/cm².

Résumé of Recent Research

The Structure of Liquid Mercury

Many liquids have now been investigated by means of x-ray diffraction techniques similar to the Debye-Sherrer system for crystals, in attempts to determine the average arrangement of the atoms in the liquid. Monatomic liquids are particularly interesting since they may be treated more exactly and should, in general, give simpler results.

Campbell and Hildebrand[1] have investigated the structure of liquid mercury using rocksalt monochromated Mo $K\alpha$ radiation. Patterns were obtained under atmospheric pressure at $-38°$, 0°, 50°, 100°, 150°, and 200° C. The x-rays were diffracted by a horizontal mercury surface at an angle of incidence of 4°45′ and recorded on a quadrant of photographic film a distance of 7.00 cm from the sample.

Analysis of the intensity patterns showed that there are approximately 6 (6.00 to 5.25) atoms in the first coordination layer of the liquid at an average distance of 3.00A from the central atom. At the lower temperatures a marked second coordination layer of 5 to 6 atoms was found in the neighborhood of 4A. This second shell gradually disappears as the temperature is raised. Further concentrations were observed at distances of 6A and 7A from the central atom.

The first coordination layer corresponds closely to the arrangement in the solid where 6 atoms occur at 3.00A with 6 more at 3.4A. This liquid coordination number of 6 is considerably lower than reported values for other liquefied monatomic elements which may account for some of the very unusual properties of liquid mercury.

As has been predicted from the general conception of liquid structure the average distance to the first coordination layer varies only slightly with temperature and the distance of closest approach of the atoms decreases with rise in temperature. The disappearance of structure with rise in temperature is also shown in the original paper where the atomic density is plotted as a function of the distance from the central atom.

[1] J. A. Campbell and J. H. Hildebrand, J. Chem. Phys. 11, 330 (1943).

New Books

Physics and Philosophy

By Sir James Jeans. The Macmillan Company, New York, 1943. Price $2.75.

Sir James Jeans is well known to physicists for treatises on mechanics and on electricity and magnetism, to astrophysicists for work on stellar dynamics, and in recent years to a wider public through a series of books discussing some of the implications of modern developments in physics. In this latest book he gives an account of the objectives of physics and of philosophy, and of the way in which these subjects have interacted as they have come in contact at various stages in their historical development. It is not a book of brilliant epigrams but is a sober and instructive account of the bearing that classical physics has had, and that modern physics may have, on various philosophical attitudes.

In general, the attitude of physicists toward philosophers has been one of rather contemptuous superiority. It has often been suggested that philosophers use a multiplicity of words to conceal a scarcity of ideas. Indeed, Jeans' account of numerous philosophical arguments tends to justify such an attitude. It seems almost incredible that such childish statements and unconvincing reasoning should have occupied the attention of serious thinkers and have been the object of widespread interest and discussion. And yet they were and to some extent still are. Jeans suggests the explanation.

The classical philosophers were concerned with the formulation of "scientific" proofs for the articles of religious faith that dominated their times, and present-day philosophers are concerned with the equivalent development of a philosophy of life. To this end, such problems as that of free will are of major importance, and are generally not only entirely unsolved but are not yet at all adequately formulated. Not only does a philosopher not know whether men have freedom of will or not, but he does not have a clear idea of what he is talking about when he mentions it. Physicists have excluded such questions from their professional consideration but have often conducted their practical affairs as though freedom of the will were self-evident.

On the other hand, philosophers have often discussed at great length the discoveries of physics with only a very superficial knowledge of them. As Jeans points out, the significant part of a physical theory is its mathematical formulation, while as a rule the philosophers have been concerned with, and often only acquainted with, the verbal description that grows up around the mathematical core.

After a description of some of the results of the theory of relativity and of various formulations of quantum mechanics, Jeans proposes a rough outline of what might be regarded as the nature of the "real world." In addition to the usual dualism of mind and matter he proposes an additional dualism of phenomena of which the mind can be conscious and a substratum entirely beyond the reach

of consciousness. This substratum, according to Jeans, is made necessary by the facts of quantum mechanics.

One can raise a great many objections to such a formulation; but whether it or any modification of it will prove satisfactory must depend on whether it is of any real assistance in formulating philosophical problems more satisfactorily than has been done in the past. Certainly, the philosophical problems cannot be ignored. They represent an important part of man's effort to adjust himself to his surroundings and they always demand solution.

W. V. Houston

Electrical Counting

By W. B. Lewis, Pp. 144, Figs. 71, 15×22 cm. University Press, Cambridge, and Macmillan Company, New York, 1942. Price $2.50.

This little book should be read by every serious student of experimental physics and should be studied by every nuclear physicist and electronic circuit man. No matter how experienced an individual may be in the field covered by this book he is practically certain to pick up at least one worthwhile idea which will be new to him. Its content is broader than implied by its modest title *Electrical Counting with Special Reference to Counting Alpha and Beta Particles*. The treatment is clear, succinct, and logical. Substantially every circuit problem involved in the design, construction, operation, and servicing of various types of counters is discussed. The apparatus required for such counting experiments includes common circuit elements such as voltage stabilizer circuits, feedback amplifiers, cathode followers, mixing circuits, triggered circuits, and discriminators. Inclusion of this material makes about three-fourths of the book useful and practical to all circuit designers. In addition to this ground work, professional nuclear physicists will find helpful hints on the design of ionization chambers, the operation of discharge counters, recording oscillographs and message registers, the statistics of counting random distributions, and a brief but valuable summary on alpha-ray energy determinations by means of range measurements.

Dr. Lewis devotes a section to each of the fundamental components of detection apparatus. Each such section considers one or two basic circuit elements which experience has shown come closest to performing the desired functions. Occasionally Dr. Lewis commends a particular circuit as almost perfect in performance, but in every case the treatment is highly critical; the disadvantages, advantages, and alternatives receive proper consideration.

Almost nothing in the book is second hand. Dr. Lewis has battled the circuits he describes, and the inventors of particular circuits will usually find their offspring described, criticized, and commended from the viewpoint of Dr. Lewis' own experience with the circuits.

Specific circuit constants are generally based on vacuum tubes available in England before the war. American workers should find that Dr. Lewis' clear discussion of fundamentals will permit them better to understand their own pentode-studded circuits, although the triode predominates in British usage. No circuit developments or improvements made since the war began are included, but as a summary of principles and practice up to about 1929 the book has undoubted permanent value.

Robley D. Evans

Here and There

Hospital X-Ray Uses Turntables

An x-ray machine incorporating a miniature turntable that enables physicians to change the position of a patient at will during delicate operations to remove foreign bodies from the lungs has been developed by the Westinghouse Electric and Manufacturing Company for the University of Pennsylvania Hospital. The x-ray machine, called a biplane fluoroscope because it enables examinations to be made in two planes—horizontal and vertical—has been installed in the hospital's department of radiology. It will be used to locate such foreign bodies in the lungs as coins and safety pins. The turntable makes it possible to position a patient so that x-ray pictures can be made from any angle, even vertical, without lifting or turning him bodily. The result is greater safety and comfort for the patient, greater usefulness for the biplane fluoroscope.

★

Dr. Slepian gets Lamme Medal

The Benjamin Garver Lamme Medal, awarded annually by the American Institute of Electrical Engineers for meritorious achievements in development of electrical apparatus or machinery, was presented recently to Dr. Joseph Slepian, associate director of the Westinghouse Research Laboratories. The gold medal, established in 1924 by the will of Mr. Lamme, chief engineer of the Westinghouse Electric and Manufacturing Company and an associate of Dr. Slepian, was presented to the winner by Harold Osborne, A.I.E.E. president. The ceremony took place during the five-day national technical meeting of the A.I.E.E. at the Hotel Cleveland.

★

Floating Lights Aid Seaplane Landings

Because water absorbs rather than reflects light, it cannot be flood-lighted for airplane landings like a land runway. But floating rubber "doughnuts," carrying intense, battery-operated fluorescent lamps, now clearly define water runway areas, permitting safe landings even on the darkest night. For these floating beacons, Westinghouse engineers developed a new hot-cathode type of fluorescent

lamp which reduces the necessary electrical starting "pressure" to only 45 volts—one-sixtieth that required for the special type of cold-cathode fluorescent lamp formerly used.

★

Three Harvard professors, who have recently visited Mexico, have been elected correspondent academicians of the Mexican National Academy of Sciences at a meeting of the Academy. Those honored were, Dr. Harlow Shapley, Director of Harvard College Observatory, Dr. L. C. Graton, Harvard geologist, and Dr. P. W. Bridgman, Harvard professor of physics. This is considered an important step in furthering closer cultural relations between Mexico and the United States.

Recent Applications of Physics

Tests for Transformers Tests made at the Sharon transformer works of the Westinghouse Electric and Manufacturing Company provide the means for a study and reduction of transformer noise. Two trucks, each carrying a transformer, are wheeled into position in a room specially constructed to keep exterior sounds out. The doors are then closed and by means of instruments, located on a panel outside the vault, the hum emitted by each unit is measured. Truck contacts engage jack contacts on the inside wall of the vault; a small microphone picks up the sound, the relative value of which is read by suitable instruments.

Increase in Electrical Power A new turbine generator of 50,000-kilowatt capacity, sufficient to supply all the electrical needs of a city of 50,000 population, has just been placed in service by Consolidated Edison. Some of the parts for this turbine were produced by the power company in order to complete the machine when pressure of other war work halted its construction in a Westinghouse plant. Because of the high quality of the machining operations performed by Consolidated Edison, the Westinghouse Company subsequently let sub-contracts to its customer for the machining of turbine spindles, gear cases and bearings, and a volume of detail turbine parts for the Naval shipbuilding program. According to J. H. Spraggon of Westinghouse, this industrial teamwork provides more electrical power for war industries in the New York area.

Generator Brushes for Aircraft A new method of treating the carbon brushes for aircraft generators has been developed by Dr. Howard M. Elsey, consulting chemist of the Westinghouse Research Laboratories. This development offsets the effects of thin air and low humidity at high altitudes and life of the brushes has been increased from an average of two hours to well over 100 hours for high altitude flying over 30,000 feet. These new brushes are now being applied in bombers and fighting planes. They were devised after extensive laboratory tests in which Dr. Elsey observed brush characteristics in a vacuum jar and in specially designed chambers simulating high altitude conditions. It is now expected that further developments, still on test, will provide even longer life than has already been made possible.

Aircraft Stamping Dies To meet the demand for faster production of the multitude of blank and pierce dies needed by the aircraft and other industries, a 300 percent faster method of manufacturing such dies has just been announced by Algoma Products, 3080 E. Outer Drive, Detroit, Michigan. In addition the process reduces the cost 65 percent. In the Algoma method, dies are produced from templates of the parts themselves, eliminating the need for special die design. The construction employed simplifies alterations due to engineering changes. The usual complex, time-consuming, and costly milling operations are replaced by a method in which dies are contour-sawed from chrome molybdenum steel, filed, sheared to size, hardened, and welded to inexpensive back-up plates. One-third of the critical alloy steel required in conventional dies is thus conserved. Individually replaceable pierce punches of hard drawn tool steel are pressed into the blank die and matching holes drilled in the blank punch for piercing holes in the part.

These dies are self-stripping and self-ejecting. All punches and dies are treated to withstand continuous production service. The punch and die are kept in alignment by two guide pins of different diameters in one portion of the die and matching hole in the mating part; the different diameters prevent incorrect assembly of the dies in the press.

Electronic Diffraction Camera Dr. Earl A. Gulbransen of the Westinghouse Research Laboratories is investigating the atomic structure of coatings that form on steel, aluminum, and copper when these metals are exposed to air or corrosive chemicals. Electrons are shot through the three-foot long vacuum tube and bounced off a highly polished button of aluminum or steel on which an oxide coating is being built. The electrons ricochet off the faces of the block-like molecules that form the coating and con-

tinue downward at an angle to strike a strip of photographic film. The electrons trace a pattern of black and white semi-circles that appear when the strip is removed from the camera and developed. This design is formed by the electrons bouncing off the different faces of the molecules in the coating.

The electronic bullets are forced out of a small aluminum rod at the top of the vacuum tube by 50,000 volts. When the voltage is applied to the rod, a pinpoint area on the end of the rod gives off free electrons that speed through the tube at the rate of four and a half million miles an hour. The polished metal buttons that Dr. Gulbransen uses as targets are inserted six at a time in a magazine that fits into the side of the vacuum tube. By revolving the magazine he can bring any of the buttons into the field of electronic fire. Electric resistance wires in the magazine heat the buttons to accelerate the formation of coatings. Growth of the oxide coating can be also speeded by admitting measured amounts of oxygen and hydrogen into the tube.

Dr. Gulbransen's work with an electronic diffraction camera is pointing the way toward longer-wearing bearings and cylinders for airplane and automobile engines, better tin plate in which the tin will cling more tightly to the steel underneath, and cheaper methods of making stainless steel.

Walkie-Talkies The War Emergency Radio Service Fact Sheet gives information regarding the small portable sending and receiving sets carried by the defense forces. By means of these sets, called Walkie-Talkies, an incident officer, or a fire chief, or chief of any emergency unit can direct his squads easily and quickly from a vantage point at the scene of disaster.

The new emergency radio system is of direct use to many strategic centers and installations in the community. Its importance appears in the two following representative cases:

1. *Industrial plants.*—Calls to an industrial plant give air raid warning, advise of latest developments during a raid, confirm calls for emergency units and indicate the help coming, and advise in the operation of the plant's own emergency forces. Calls from an industrial plant summon emergency medical teams, rescue units, fire and police forces, demolition squads, and if needed, decontamination units. Large industrial plants can use walkie-talkies to reach plant protection volunteers in outlying sections of the plant or in separate buildings or defense posts.

2. *Hospitals.*—It is crucial to know during a raid exactly what beds are available and what operating rooms are free in the casualty receiving hospitals of a community, and to direct casualties rapidly to available facilities. Ambulances can be loaded and dispatched effectively because the control center is in communication both with hospitals and with the incident officers and incident medical officers. Mobile medical teams can be directed from point to point without returning to their bases. If the hospital facilities of an area become overtaxed, the radio channel to the district headquarters can arrange for reinforcing hospital facilities.

WALKIE-TALKIE GOES TO SEA

Although developed to ride a soldier's back in steaming jungles, desert wastes, or arctic drifts, the "walkie-talkie," one-man radio station built by Westinghouse for the Army, now has gone to sea. Its ten-mile range is short enough so that ships in convoy can transmit voice messages from vessel to vessel without using the "big" radio, which might give away ship location to enemy listeners.

Innovations in Instruments

New and Rare Instruments

Information concerning the following offers and requests for rare instruments that can be sold, loaned, or leased for essential war research work can be obtained from D. H. Killeffer, Chairman of the Committee on Location of New and Rare Instruments, 60 East 42nd Street, New York 17, New York.

Instruments Offered:

L & N Thermionic Amplifier; Ammeters—Whitney and Jewel; Various Balances; Schmidt & Haensch Colorimeter; Platinum Calorimeter; Weston D. C. Electric Meters—Voltmeters, Ammeters, Galvanometers; Two-Circle Goniometer (two loan for war use); Microscopes—Bausch & Lomb Centrifuge, Zeiss Binocular, Leitz (suitable for photomicrography); Poeller Optical Wedge Pyrometer; 4 L & N Portable Potentiometers; L & N Type *K* Potentiometers; Polarizer and Analyzer for microscope; Abbé Refractometer; R. Fuess Reflectometer; Spectrographs, Spectroscopes, Spectrophotometers; Saccharimeters; Saybolt Universal Viscosimeter.

Instruments Requested:

Western Electric Electrometer; Flexaform Set; Amsler No. 4 Integrator; Micromanipulators.

Student's Kelvin Bridge

A new illustrated 8-page booklet issued by the Leeds & Northrup Company describes the Student's Kelvin Bridge [Catalog E-53(1)], an instrument which provides the features which teachers have found most valuable in imparting an understanding of the Kelvin bridge method. Although intended primarily for instruction, the bridge is said to be suitable for general use in the measurement of resistance from 0.1 to 0.001 ohm, when only moderate accuracy is required. Its design is simple, and affords a clear illustration of the fundamental principles of the method. Provision is made for inserting extra resistances in either of the ratio circuits or in the yoke, so that the effects of resistance in these parts of the bridge can be demonstrated. Copy of this booklet sent upon request by Leeds & Northrup Company, 4934 Stenton Avenue, Philadelphia, Pennsylvania.

Contributed Original Research

The Approximate Mathematical Methods of Applied Physics as Exemplified by Application to Saint-Venant's Torsion Problem

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(Received July 7, 1943)

THIS paper is one of a series of three papers devoted to exposition of the known solutions of Saint-Venant's torsion problem for a prism of *uniform cross section*. The first,[1] published some time previously, comprises account of the historical development of the fundamental mathematical and physical theory of the torsion problem and of the solutions that have been effected by exact analytic methods. The third,[2] to be published shortly, covers the solutions that have been effected by empiric and by experimental methods. The present paper, the second of the series, embraces those solutions that have been effected by various approximate methods.

The more important of these methods comprise three categories: variational, arithmetic, and graphic—according as they stem from the theory of the calculus of variations, from the theory of linear difference equations, or from the use of graphic analysis. In that they afford, in divers branches of elastic theory, solutions of problems mathematically intractable to exact analysis (which category includes, for example, most of the important practical torsion problems associated with the stress analysis of high speed airplane, engine, and tool designs) these methods are of considerable technical importance. Yet, despite this significance, no considerable exposition of these approximate methods is to be found in standard reference texts on elastic theory or applied mathematics; nor for that matter, is other than limited mention* made therein of sources, either original or secondary, that contain such exposition.

A host of such sources, however, is mentioned in this paper. And in that these furnish a connected and logical introduction to a number of the more powerful approximate mathematical methods available for effecting actual numerical solutions of *two-dimensional equilibrium* problems in all branches of applied physics, especial interest may attend the following inclusive, and fully documented, account of the application of these approximate methods to Saint-Venant's torsion problem.**

I. VARIATIONAL METHODS†

1. Ritz's Method and Modifications

Thirty-five years ago (1908) the Swiss physicist Walther Ritz published a memoir[3] wherein he propounds a new method of solving problems of mathematical physics; and in this and subsequent memoirs,[4,5] he effects by this method approximate solutions of problems which had long defied the efforts of analysts. Among the classical problems treated in the first memoir is the Dirichlet problem for a singly-connected region. Inasmuch as the torsion problem for a

* In fact, none whatsoever of most of the methods discussed in this paper.

** For a qualitative account of the torsion problem see reference 1; for accounts of the general mathematical theory see: S. Timoshenko, *Theory of Elasticity* (McGraw-Hill Book Company, New York, 1934); I. S. Sokolnikoff, *Mathematical Theory of Elasticity* (Brown University, Providence, 1941); A. E. H. Love, *The Mathematical Theory of Elasticity* (University Press, Cambridge, 1934), fourth edition.

† The notation and terminology of this paper are those of Love: μ denotes the modulus of rigidity; τ the angle of twist per unit length; and the other symbols are defined as introduced. To avoid redundance symbols are used in the text in lieu of the corresponding phrase. Again, inasmuch as determination of either the torsional moment M or the torsional rigidity C is tantamount to determination of the other, for $(M=\tau C)$, mention is made only of M—though in some of the papers discussed it is the expression for C, and not M, that is given.

"singly-connected" prism†† is interpretable as a first Dirichlet problem (namely, as the search for the conjugate torsion function ψ that satisfies $\nabla^2\psi=0$ over the right cross section S of the twisted prism and satisfies $\psi=(x^2+y^2)/2+a\ constant$ on the boundary s of S) the solution for a particular cross section is afforded by Ritz's method—sometimes termed the Ritz-Rayleigh method, Lord Rayleigh[6] having pointed out that Ritz's procedure has much in common with approximate methods used earlier by him.

Application of Ritz's method to a problem in elasticity hinges on expressing the problem as one in the calculus of variations, a transformation usually effected with ease inasmuch as the definitive partial differential equations of most physical problems can be formulated from one of the well-known "principles" of physics—for example, Hamilton's, or Maupertuis', or Castigliano's principle. Transformed, the problem reduces to determination of a function which minimizes a certain integral expression subject to specified boundary conditions. The torsion problem, for instance, can be interpreted as the search for the stress function $\Psi[\equiv\psi-(x^2+y^2)/2]$ that minimizes the integral expression

$$U(\Psi)=(\mu\tau^2/2)\int_S \text{grad}^2\,\Psi dS$$

for the elastic energy per unit length $U[\equiv\tau M/2]$ subject to the conditions: (i) the twisting couple

$$M=2\mu\tau\int_S \Psi dS$$

is a prescribed constant; (ii) $\Psi=a\ constant$ (commonly taken as zero) on s.

An alternative, somewhat more convenient statement of the problem is to find Ψ that minimizes the integral expression*

$$I(\Psi)=(\mu\tau^2/2)\int_S (\text{grad}^2\,\Psi-4\Psi)dS$$

subject to the boundary condition $\Psi=0$ on s. And yet other forms of the integral expression to be minimized by the same Ψ are to be found.

The tie between the two methods of interpreting the torsion problem—as a variational problem or as a boundary-value problem in partial differential equations—lies in this: The Ψ which solves the variational problem necessarily satisfies** a characteristic partial differential equation (also termed the Euler equation); this equation is precisely the partial differential equation occurring in the boundary-value problem; and inasmuch as the boundary condition is the same in both problems—necessarily so,*** in fact—it follows that this Ψ is also that which solves the boundary-value problem.

In general, Ritz's method does not yield the function Ψ that exactly minimizes the integral expression $I(\Psi)$: Rather, it yields an nth approximation Ψ_n to Ψ—where Ψ_n is expressed as a linear combination of n functions of a certain type individually satisfying the boundary conditions—and an nth approximation $I_n(\Psi_n)$ to I. But *often* the approximation to the true minimum I can be effected to any desired degree in a comparatively simple manner; the approximation to the true minimum constantly improves as the terms of the assumed series increase in number; and if an appropriate selection of functions has been made—commonly, this selection is determined by experience, by the particular coordinate system employed, and by the fact that the integrand stemming from substitution of Ψ_n must be mathematically tractable—the approximation is often quite good when the number of terms in Ψ_n is small. In general, the security for Ψ_n and I_n being actual approximations to Ψ and I is ensured by satisfaction of the conditions that stem from setting the problem as one in the calculus of variations. Specifically, exact conditions on Ψ sufficient to ensure the approach of Ψ_n and I_n to Ψ and I have been the subject of

†† The torsion problem for a prism of multiply-connected cross section is also solvable by Ritz's method (see reference 15); however, the boundary conditions—more complicated than for the "singly-connected" prism—are such that characterization as a first "Dirichlet problem" is hardly proper.

* Note that by virtue of the quantities defined in the foregoing paragraph $I(\Psi)=U(\Psi)-M\tau$; and since M is a prescribed constant, $I\equiv-\tau M/2=-U$, wherein I and U designate the minimums of $I(\Psi)$ and $U(\Psi)$.

** See A. R. Forsyth, *Calculus of Variations* (University Press, Cambridge, 1927), p. 551.

*** See reference 15, pp. 4–7 and the extension of this discussion in reference 35, Chapter 3. Also, H. Bateman, *Partial Differential Equations of Mathematical Physics* (University Press, Cambridge, 1932), p. 157.

research by Ritz, by Trefftz, and by others.† Provided these sufficient conditions are satisfied, they have shown that

$$\lim_{n\to\infty} \Psi_n = \Psi; \quad \lim_{n\to\infty} I_n(\Psi_n) = I; \quad \text{and} \quad I_n(\Psi_n) \geq I.$$

Now, though Ritz's method yields a solution where classical methods fail, and though by virtue of the inequality $I_n \geq I$ this solution is an upper limit to the exact solution, it commonly ensues that the degree of approximation to the exact solution is unknown. In appreciation of this difficulty Trefftz has developed a method, akin to Ritz's, but so designed that $I_n'(\equiv I_n)$ approaches the true minimum I from below. As with Ritz's method, the approximations to Ψ and to I can be carried to any degree of accuracy; and providing Ψ_n satisfies certain known conditions,

$$\lim_{n\to\infty} \Psi_n = \Psi; \quad \lim_{n\to\infty} I_n'(\Psi_n) = I;$$

but now $I_n'(\Psi_n) \leq I$. Thus, by application of both Ritz's and Trefftz's methods the true minimum is bounded in known fashion, $I_n' \leq I \leq I_n$; and the sign and magnitude of the maximum possible deviation of I_n, I_n', or their arithmetic mean (possibly a closer approximation) from I is determinate.

Trefftz has developed the theory of his method in two memoirs. One,[7] essentially mathematical, is devoted to elaboration of the general theory of his method. The other,[8] essentially expository, comprises solution of the torsion problem for certain cross sections: Ψ_1 and M_1 of a [prism of] square [cross section] are obtained by both his method and Ritz's and the two solutions compared in various fashions; a device for extending both methods to provide solutions for cross sections having re-entrant angles is outlined; by both methods M_1 is effected for an angle-iron with identical, finite flanges.

Consideration of solutions effected by both methods indicates that, in general, certain advantages attend the use of Trefftz's rather than Ritz's method. Thus—as Trefftz professes—although the approximating functions used in the latter need not necessarily satisfy the partial differential equation defining the problem, they must satisfy the boundary conditions—not always a simple thing to ensure. Contrariwise, Trefftz's method—to be characterized as a generalization of the classical method of development in terms of particular solutions—requires only a choice of functions that satisfy the partial differential equation, commonly a simpler demand. Secondly, the choice of functions in Trefftz's method is usually such that determination of the maximum error in the solution is easier. Finally, proof of convergence of the approximating function to the true minimum is generally simpler. One considerable advantage, though, attends Ritz's method: ordinarily, the approximation converges somewhat faster than in Trefftz's method. Thus, Holl and Anderson,[9] using both methods to obtain values of M_2 for the cross section bounded by arcs of two confocal parabolas,* find that though both values are in good agreement with the known exact solution the value afforded by Ritz's method is in closer agreement.

A second method of obtaining a lower bound to the minimum function I solving a given problem, one in some ways more general than that of Trefftz, has been outlined by Friedrichs.[10] It consists of associating with the given minimum problem the auxiliary problem of finding the maximum of a second integral, obtained from the first by a simple transformation and of such nature that its maximum is identical with the minimum of the first integral. As, obviously, the negative of the maximum sought is a minimum, it can be obtained in a fashion akin to that of Ritz. The actual procedure can differ in that if only the approximation I_n to the extremum I is sought, and not, as well, the approximation Ψ_n to Ψ producing I, the approximating functions need not *necessarily* satisfy the boundary conditions as they must in the Ritz process. On the other hand, convergence to the minimal solution is somewhat slower.

The explicit theory essential to application of

† For a number of references to these purely formal investigations see the bibliography of reference 15; see also N. Kryloff and J. Tamarkin, Bull. Acad. Sci. Russ. [6] **12**, 69–88 (1918); E. Trefftz, *Handbuch der Physik* (J. Springer, Berlin, 1928), Vol. 6, pp. 130–132.

* Statement of an nth approximation implies determination of the preceding $n-1$ approximations.

Friedrichs' method has been developed by Friedrichs himself and by Basu.[11,12] The latter treats in detail an important topic barely touched by Friedrichs: namely, determination of the [torsion] function ϕ, the complex conjugate of ψ. If Ψ_n is obtained by Ritz's method, ψ_n follows immediately from the relationship $\psi=\Psi+(x^2+y^2)/2$. But though ϕ is the complex conjugate of ψ, ϕ_n cannot be obtained from ψ_n by the usual methods of function theory: for the terminated series expressing Ψ_n does not, in general, satisfy the differential equation of the problem $\nabla^2\Psi=-2$, even though the conditions on Ψ_n are such that the function represented by the limit of the series as $n\rightarrow\infty$ does satisfy the equation term by term (an interpretation of this latter mathematical phenomenon has been advanced by Goldsbrough[13]); whence it follows that ψ_n obtained from Ψ_n is not a harmonic function. Friedrichs merely indicates how determination of ϕ conjugate to ψ can be expressed as a variational problem. Basu develops in detail the required analysis to the end that ϕ is obtained through determination by Ritz's method of the function J maximizing a certain double integral expression. Additionally, the approximation to this integral expression furnishes the desired lower bound on I, or $J_n\leq I\leq I_n$. Further, by virtue of the identity $I\equiv-\tau M/2$, bounds are provided on M: that is, $I_n\leq\tau M/2\leq J_n$.

The details of the analysis contained in Basu's memoirs are exemplified by application to the torsion problem of the square. In the first memoir[11] Ψ_2 is effected by Ritz's method; M_2 by both Ritz's and Friedrichs' methods. In the second memoir[12] ϕ_2 is obtained by Ritz's method; and from ϕ_2 another value of M_2. Close agreement of this last value of M_2 with the known value is taken as evidence that the corresponding expression for ϕ_2 is sufficiently accurate for most purposes; i.e., to give the longitudinal displacements of points of the initially plane cross section.

At this point we must mention a certain defect in the methods discussed above. Application of Ritz's and of either Trefftz's or Friedrichs' method enables estimation of the error in an approximation to the minimum I of the integral expression $I(\Psi)$; for these methods furnish upper and lower bounds on I. Often, however, what is desired, yet usually cannot be obtained inasmuch as the mathematical technique is lacking, is knowledge of the error in the approximations afforded by Ψ_n to Ψ or to its derivatives. In the torsion problem, for instance, the longitudinal displacement and the stress distribution, respectively furnished by ϕ and by the derivatives of Ψ, are of greater immediate value in practical design than is either U or M afforded by I. But though, as just remarked, a general technique is not available for calculating the error in Ψ_n, it happens that a specialized technique applicable to the torsion problem is known.

In a paper devoted to exposition of his "thickness parameter" method, explained in Section II below, the English aeronautical engineer W. J. Duncan[14] outlines a method* of calculating the maximum deviation between the values of a function Ψ, defined by the conditions that it satisfy $\nabla^2\Psi=-2$ over S and vanish on s, and those of an approximation Ψ_n. Thus, if ϵ' and ϵ'' are the maximum and minimum values within S of a function ϵ defined by $\nabla^2\Psi_n+2=\epsilon$, then $(1-\epsilon''/2)=\Psi_n/\Psi=(1-\epsilon'/2)$; and conjunction of this knowledge with the maximum values of ϵ *on the boundary* enables determination of the maximum deviation of Ψ_n from Ψ at any point within or on the boundary of S.

This procedure, however, does not furnish—at least directly—information on the derivatives of Ψ_n; nor on ϕ, the conjugate of

$$\psi[\equiv\Psi+(x^2+y^2)/2].$$

Lacking means to obtain such information, it is tempting to conjecture that if I is bounded closely by J_n and I_n, the corresponding expressions for ϕ_n, Ψ_n, and the derivatives of the latter yield values in good agreement with the exact values. Though such inference is as yet unsubstantiated by rigorous analysis, empirical support might derive from comparison of values so obtained with exact values calculated from known rigorous solutions (see reference 1 for discussion of more than a 100 such solutions).

In a recent paper whereof several pages are devoted to a qualitative discussion of the fundamental concepts of the Ritz process and of its

* Obviously, this method is directly extensible to the more general equation, $\nabla^2\Psi=$ *an arbitrary constant.*

merits and defects as regards actual use—which discussion complements foregoing remarks—Courant,[15] illustrative of his remarks on the solution of the torsion problem for multiply-connected cross sections, obtains Ψ_2 and M_2 for a hollow square. This, to the writer's knowledge, is the only published treatment of the numerical solution of a multiply-connected cross section by Ritz's method.

Finally, we note two other applications of the Ritz analysis to the torsion problem. Approximating the normal cross section of a propeller blade by the area bounded by arcs of two non-orthogonal parabolas, Leibenzon[16] obtains Ψ_1 and a first approximation T_{m1} to the maximum shearing stress T_m; for an area bounded by one of the parabolas and by a line segment normal to its axis, Ψ_2 and T_{m2}; and for a streamline section akin to an airfoil, Ψ_2, M_2, and T_{m2} (however, T_{mn} is obtained from Ψ_n by differentiation, in which connection see the second preceding paragraph).

Duncan, Ellis, and Scruton[17] utilize Ritz's method to investigate the torsion problem for the general isosceles triangle, the exact analytic solution of which is as yet unknown, though certain special cases have been so solved. Ψ_1 and M_1 are effected. It is found of the former that it is exact for the special case of the equilateral triangle and is a good approximation for isosceles triangles with small included angles, except, possibly, near the base of the triangle; and of the latter that it is "almost certainly amply accurate for practical purposes."

2. Galerkin's Method

In 1915 Galerkin[18] advanced a variational process which has found considerable use in divers fields of applied mechanics, especially in aircraft stress analysis. Galerkin's method has much in common with Ritz's process: indeed, Biezeno,[19] demonstrating that through analytic transformation the general equations of one can be derived from those of the other, maintains that in essence they are identical. Elsewhere, discussing the relative merits of Ritz's, of Galerkin's, and of a third method advanced by Koch[20] and himself, Biezeno[21] indicates a process that encompasses the just-named as special cases. More inclusively, Gross[22] has shown that these procedures and others are derivable as special cases of a very general functional method.

In Galerkin's method an nth approximation to the rigorous solution of a one-dimensional problem is taken in the form

$$\sum_{i=1}^{n} c_i f_i(x),$$

wherein the c_i are arbitrary constants and the f_i a selected set of linearly independent functions individually satisfying the boundary conditions. Substitution of this expression in the defining differential equation (arranged with right-hand member zero) yields "a remainder" or "error" ϵ, a linear function of the n arbitrary constants. Then multiplication of ϵ by each of a set of n arbitrarily chosen, linearly independent functions termed multipliers; integration of each of the products thus formed over the region S of the problem; equation to zero of the resulting expressions; and solution of the resulting set of equations yield a preferred set of values for the arbitrary constants.

From the standpoint of mechanics ϵ is interpretable as a generalized force, the arbitrary constants as a set of generalized coordinates, the appropriately chosen multipliers as virtual displacements corresponding to increments of each of these generalized coordinates, the integrated products as weighted means of ϵ, and the vanishing of these weighted means as the vanishing of the virtual work due to the corresponding displacements. In light of this mechanistic interpretation it follows that the degree of accuracy obtained can be increased indefinitely by increasing the number of functions employed; that, in general, exact solution of an m-dimensional problem requires an m-fold infinite set of functions $f_i(x_1, \cdots, x_m)$; but that if the functions and the multipliers are chosen appropriately, it can be anticipated that a good approximation stems from use of only a few functions.

Now in terms of the familiar membrane analogy to the torsion problem Ψ is proportional to the vertical displacement of the membrane and ϵ is proportional to the applied pressure not balanced by the tensions in the membrane. Ac-

cordingly, we represent the stress function by

$$\Psi = \sum_{i=1}^{n} c_i f_i(x, y),$$

these functions f_i appropriately chosen and satisfying the boundary condition $\Psi = 0$ on s; substitute Ψ in $\nabla^2\Psi + 2 = 0$; obtain the "error,"

$$\epsilon = \sum_{i=1}^{n} b_i c_i,$$

wherein the b_i are known constants; choose as multipliers the f_i; integrate each of the products $\epsilon f_i(x, y)$ over S; equate to zero each of the resulting expressions—hence, impose the condition that the work done by the unbalanced load in a virtual displacement of the membrane proportional to $f_i(x, y)$ be zero; and solve the resulting set of algebraic equations for the values c_i.

The most inclusive treatment* in English of Galerkin's method is encompassed in the writings of Duncan.[23–27] Illustratively, Duncan[23,25] applies this theory to a variety of problems, among them the torsion problem. Thus, for an airfoil bounded by the loop of a bisymmetrical cubic oval Ψ_2 and M_2 are effected and compared with corresponding values obtained earlier by Duncan by his "thickness parameter" method. Analysis of a bisymmetrical parabolic lenticular cross section is carried out by both methods: Ψ and M are calculated, respectively, to the 4th and 6th powers of the thickness parameter; the latter is found to be in close agreement with M_2 effected by Galerkin's method. Ψ_3 is determined for a rectangle; values of M_3 for squares are in good agreement with values obtained from Saint-Venant's exact solution. Finally, Ψ_1 and M_1 are obtained for an arbitrary area possessing at least one axis of symmetry; corroboratively, M_1 yields the known exact expression for the ellipse.

3. Other Variational Methods

A variational method proposed by Poritsky[28] is of considerable usefulness. As exemplified by the torsion problem Ψ is taken to be of the form

$$\Psi = \sum_{i=1}^{n} X_i Y_i,$$

wherein the Y_i are assumed functions of y alone and are such that Ψ satisfies the boundary condition $\Psi = 0$ on s, and the X_i are unknown functions of x alone. Then substitution of Ψ in $I(\Psi)$ and imposition of the variational condition that $\delta I(\Psi) = Y_i \delta X_i$ vanishes for all δX_i yields a set of n differential equations, each containing but one X_i. Whence the set of X_i is determined by finding a set of solutions that satisfy boundary conditions determined in part by the assumed form of the Y_i and in part by the original boundary condition.

In this fashion Poritsky obtains Ψ_1, and from it M_1 and T_{m1}, for a certain rectangle. These values are in excellent agreement with corresponding values calculated from Saint-Venant's exact solution. In similar fashion Ψ_1 is obtained for the ellipse and for the rhomboid.

In solution of the torsion problem, or yet other two-dimensional problems pertinent to a given area, certain modifications of the above procedure may prove advantageous: say, interchange of the values of X_i and Y_i; or choice of Y_i as a function of both x and y; or utilization of some other permissible variant.

Yet another variational method is described by Hovgaard.[29] By conjunction of Castigliano's principle and the use of Lagrange's multipliers general variational equations are derived for a prism constrained to simultaneous flexure and torsion. Though the details of solution are not given, Hovgaard states that these equations have been used to solve the torsion problem for the ellipse, quasi-ellipse, equilateral(?) triangle, and rectangle.

II. DUNCAN'S METHOD OF "THICKNESS PARAMETER"

Of especial interest in aircraft stress analysis is solution of the torsion problem for uni- or bisymmetrical cylinders of cross section such that all values of the half-thickness t are small compared with the length of the longest (major) axis of symmetry. Particular interest, therefore, accrues to methods especially adapted to the solution of such thin cylinders. Possibly the most effective is Duncan's[14] method of "thickness parameter."

Directing the x axis of a Cartesian system along the major axis of the cross section, the boundary equation is written $y \equiv t = \theta T$, wherein

* See also a very brief discussion in E. Keller, *Mathematics of Modern Engineering* (John Wiley and Sons, New York, 1942), Vol. 2, pp. 288–292.

θ is the thickness parameter independent of x and T is an assigned function of x. Now, by virtue of symmetry about the x axis only even powers of t and y can appear in Ψ; further, by virtue of the boundary condition on Ψ it vanishes on s. These considerations suggest expression of Ψ in the form

$$\Psi = t^2 - y^2 + \sum_{i=1}^{\infty} (t^{2n} - y^{2n}) f_n(x), \quad t \equiv \theta T,$$

wherein the functions $f_n(x)$ are to be determined from the condition that Ψ satisfies $\nabla^2\Psi = -2$. The $f_n(x)$ determined—discussion of the actual procedure is out of place here—Ψ appears as a convergent series in powers of θ^2 with coefficients which are functions of x. Commonly, Ψ is determined only to some specified power (say, the kth) of θ; hence, $\Psi_k \sim \Psi$. With Ψ_k determined, the torsional moment and the shearing stress are obtained from the known relationships in which, however, Ψ_k replaces Ψ—though no proof of validity of this procedure is advanced. Nor is proof given that

$$\lim_{k \to \infty} \Psi_k = \Psi:$$

though inasmuch as the procedure (Duncan's) discussed in Section I:2 affords calculation of the maximum error in Ψ_k, this defect is of academic rather than of practical significance.

Illustrative of his analysis Duncan obtains the exact form of Ψ for the ellipse, equilateral triangle, thin isosceles triangle with two infinite sides, and curvilinear isosceles triangle comprised of two linear sides and base a hyperbolic arc subtending 45°. An area bounded by the loop of a cubic curve is treated at length: Ψ_9, M_{10}, and S_7 are obtained; a detailed calculation of maximum error in Ψ_9 is set out.

In a succeeding paper Duncan[30] extends the method of thickness parameter to doubly-connected and to asymmetrical cross sections. A very complete discussion of the general theory is illustrated by determination of approximate values of Ψ and M for the slit and the complete annuli, and for a cross section bounded by concentric similar ellipses.

III. FINITE DIFFERENCE METHODS

The theory of finite differences affords powerful methods of effecting numerical solutions to boundary value problems which are mathematically intractable to attack by exact analytic methods. In one fashion the general scheme of solution involves (i) replacement of the partial differential equation and the boundary conditions defining the problem by an analogous partial difference equation and equivalent boundary conditions; (ii) effectance of a sequence of approximate numerical solutions of the difference problem; (iii) application of a proper test to determine the convergence of this sequence to the exact solution; (iv) determination of the limits of error in the final approximation. In yet another fashion the problem can be set up as a variational problem involving minimization of an integral expression, this latter paralleled by its finite difference analogue, and approximate solution of this last effected in an appropriate fashion.*

Though solution of boundary value problem by application of finite difference methods is an idea far from new, the pertinent theory is, seemingly, but little known among applied physicists, and even less so among engineers. In part, at least, this state of affairs is attributable to the lack of a suitable reference text in English. Thus, though L. M. Milne-Thomson's *The Calculus of Finite Differences* covers the formal mathematical theory of finite differences, an equally authoritative text on *applied* finite-differences (aside from actuarial use) is yet to be written. Perforce, then, the earnest worker must turn to the periodical literature on applied finite differences. Hereof, a logical course of study might well encompass reading of the work of, say, Richardson,[31,32] Liebmann,[33] Wolf,[34] and Courant[35]—who present general theory from a semi-applied point of view—and subsequent study of a sequence of papers wherein development of theory is illustrated by application to one of the classical problems of the clamped plate, the membrane, or the torsion problem.†

* A good illustration of this second kind of analysis is contained in S. J. Newing, Phil. Mag. **32**, 33–49 (1941). Incidentally, the analysis of reference 31 is extended herein.

† Of this literature no considerable published bibliography is known to the writer. Of his own inclusive bibliographies on the three problems mentioned, that pertinent to the torsion of uniform prisms is presented in this paper; that to non-uniform but axially symmetrical prisms is to be presented in a subsequent paper.

In a recent paper (reference 15) wherein the fundamental concepts of certain finite difference procedures are

To the writer's knowledge the first solution of the torsion problem by finite differences was effected by Runge,[36] who determined and plotted the stress lines in a cross-shaped area comprised of five squares. Extension of Runge's analysis to multiply-connected areas is discussed by Neményi,[37] though he advances no explicit solutions of the torsion problem.

In 1918, in a paper frequently mentioned in the subsequent literature, Liebmann[33] suggested certain modifications in the methods then used for solving physical problems by finite differences. Subsequently, these suggestions were reworked, improved, and extended by many writers, to the end that comparison of their papers reveals such overlapping that it is a matter of some difficulty to assign specific improvements in technique to given writers.

Gerschgorin has written a series of papers, for the most part in Russian, wherein linear difference equations are applied to effect the solution of problems involving Laplace's and Poisson's equations. Fortunately for the linguistically limited, one[38] of the more important papers is available in German. Therein, methods for calculating the error pertinent to the approximate solutions of a linear partial differential equation of elliptical type (the ubiquitous Laplace and Poisson equations are of this type) by difference processes are discussed. The theory developed in this paper is exemplified by approximation of Ψ for a specifically dimensioned rectangle, determination of the error at a selected point of this rectangle, and check of this latter through comparison with corresponding values determined from the known solution. Collatz[39] has extended the general analysis of this paper.

Schwalbe,[40] also, has studied solutions of the rectangle by finite difference methods. Numerical values for M and T_m are found to be in good agreement with corresponding values obtained from the exact solution. Additionally, finite difference analyses are effected for the torsional moments of a sharp-cornered channel of uniform thickness and an I beam with sloping sides. Numerical values calculated therefrom are compared with corresponding values obtained both from experiment and from various analytic solutions.

Certain improvements in the original Liebmann process are professed in a series of loosely connected papers by Shortley and Weller;[41] Weller, Shortley, and Fried;[42] Frocht and Leven;[43] and Poritsky, Snively, and Wylie.[44] Essentially, these papers are concerned with the development of methods that minimize the numerical labor incident to the solution of a given problem. Fuller discussion here is not feasible. The theory developed in the second of these papers is illuminated by solution of M and T for two squares with rounded corners—the sides are equal, the radii different.

In the course of the past few years Southwell[45] and a group of fellow workers have applied a numerical method, the "relaxation method," to effect the solutions of divers types of physical problems. Originally devised by Southwell for the determination of stresses in redundant frameworks—in which connection it has much in common with Hardy Cross' [46] method of distributed moments—it was subsequently developed to the end that in an investigation of the general theory of the application of relaxation methods to linear systems, Temple[47] remarks that "almost all linear systems of equations, algebraic, integral, or differential, can be brought within the scope of relaxation methods, which seem to constitute one of the most powerful methods of computation in mathematical physics and engineering." Investigations of necessary and sufficient conditions ensuring that the approximations stemming from relaxation methods converge to the exact solutions are to be found in papers by Black and Southwell[48] and by Temple.[47]

Christopherson and Southwell[49] have conjoined the relaxation process with a finite difference procedure indicated earlier by Thom[50] to effect the solution of the torsion problem for an equilateral triangle pierced by three symmetrically located circular holes. Ψ is approximated and the stress lines plotted. Preliminary to the solution of this problem and indicative of the best procedures to be followed in solving it, Ψ of the equilateral triangle is approximated.

briefly but interestingly discussed, Courant mentions a number of papers on finite differences. Many of these papers, though, are on purely formal points of the theory and contain no illustrative examples; however, taken as a third unit in the above scheme of study, they can be read with profit by those possessing the required mathematical background.

Earlier, Orr[51] had used an extension of Thom's method to effect approximations to M and T_m for a serrated annulus and for a hollow square. Additionally, approximations to M for a number of standard structural sections are determined, and then compared with experimental values.

Finite difference analysis developed by Sunatani and Negoro[52] has been employed by the latter[53] to study the stress distribution and to determine T_m in squares pierced by a centrally located circular hole. Values of the latter are in general agreement with corresponding values determined by experiment. Additionally, an estimation of accuracy is afforded by comparison of approximate values calculated for T_m in the annulus with values calculated from the known solution; and by comparing values of T_m for the pierced squares with corresponding values obtained by experiment.

IV. GRAPHIC METHODS

Industrial stress analysis frequently requires solution of the Dirichlet problem for an area bounded by a curve unamenable to simple mathematical expression. In this case recourse to some form of experimental or graphic solution may prove profitable. In this connection special interest attaches to Bairstow's[54] ingenious analytic-graphic method based on a theorem due to Green: ". . . any irrotational acyclic fluid motion can be reproduced by an appropriate choice of simple sources round the fixed boundary." As applied to the torsion problem, a process of repeated contour integration of the known boundary values of $\psi[=(x^2+y^2)/2 \text{ on } s]$ yields the strength χ of these boundary sources. Theoretically, rigorous determination of χ requires, in general, an infinite number of repeated integrations; practically, a good approximation χ_n to χ is commonly afforded by calculation of only a few terms of the sequence. χ_n determined, the torsion functions are found by appropriate integrations or differentiations, effected either analytically or graphically.

Illustrative of theory Bairstow, using graphic integration, obtains χ_2 for the ellipse, χ_9 for the eccentric annulus, and χ_3 for a cross section akin to that of a propeller blade. For none of these configurations does Bairstow prove the convergence of χ_n to χ; nor is a proof of convergence for the general case advanced. In the same paper, however, Perry (writing separately) through analytic evaluation of the integrals occurring in the solution of the ellipse and the eccentric annulus shows that in these cases, at least, the approximate solutions converge to the known exact solutions.

In a subsequent paper Bairstow and Pippard[55] advance an inclusive account of the theory of this method: Each step is justified rigorously and is interpreted physically. Illustratively, the scheme of solution and the pertinent mechanical details of the graphic analysis are delineated through solution of numerical values of χ, Ψ, ϕ, M, and T for an ellipse. Excellent agreement of the calculated numerical values for the shear stress on the boundary with those calculated from the exact analytic solution are interpreted as justification of the accuracy of the graphic procedure. Less detailed treatments are given of several configurations of practical interest: a slotted circle, a slotted eccentric annulus, and a series of squares with rounded corners. These shapes approximate the cross sections of standard types of shafts.

We note here that a decade before Bairstow advanced his method Runge[56] outlined a method of solving a Dirichlet problem by graphic integration of an integral, the integrand of which is a function of the Green's function for the problem. Though Runge discusses the general theory in detail, seemingly neither he nor others have applied it to solution of the torsion problem.

Though the most accurate of the graphic procedures available for solving Dirichlet problems, Bairstow's procedure is also the most elaborate and time consuming. Where a less accurate procedure will suffice the entirely graphic method of "curvilinear chequers" may be used to advantage. Suggested by Maxwell, it was subsequently developed by many writers. We mention here only a lengthy and exhaustive memoir by Richardson;[57] the theory therein suffices for the solution of a great variety of two-dimensional problem. In a recent paper Poritsky[58] illustrates application of this method to the torsion problem by effecting the stress plots of several cross sections.

Perkins[59] has advanced an analytic-graphic means of approximating the torsional moment

and the boundary stress. Illustratively, he approximates M for the circle, annulus, and equilateral triangle.

V. MISCELLANEOUS METHODS

Of considerable practical interest is a method suggested by Barta[60] for the solution of a problem of equilibrium that entails determination of a function vanishing on the boundary of a given two-dimensional region. As applied to the torsion problem Ψ, expressed as a Fourier-Dini series with unknown coefficients, is approximated by Ψ_n, the sum of the first n terms of Ψ. Then successive substitution in Ψ_n of the coordinates of n selected points of the boundary (whereon, $\Psi=0$); equation to zero of each of these expressions; and solution of the resulting set of n linear equations yield a set of preferred coefficients for Ψ_n. Presumably, the degree of approximation increases with n; and proper selection of the points on the boundary, bilateral or quadrilateral symmetry of the given domain, and other factors operate to afford rapid convergence to Ψ. Thus, Barta finds that Ψ for a square yields numerical values in excellent agreement with the exact values.

Useful extensions of Barta's method readily suggest themselves. For example, if the boundary of the given domain is such as to defy simple analytic description, it can be drafted and the coordinates of the selected points determined with an accuracy limited only by the scale of the drawing. Or in virtue of the shape of the given domain advantage may accrue to recasting the analysis in terms of another kind of series, say a Fourier-Bessel expansion. Or conformal mapping can be conjoined with Barta's procedure. And yet other variants are feasible.

The Fourier-Dini series has also been utilized by Csonka[61] in a method designed for approximating Ψ in a doubly connected region. Laurent's theorem enables expansion of Ψ as a sum of unknown harmonic functions; in turn, each of these functions is expressed as a Fourier-Dini series with unknown coefficients; these latter are then determined through analytic manipulation of the prevailing boundary conditions; whence Ψ is established. Often, as Csonka demonstrates by numerous examples, Ψ is closely approximated by Ψ_2 or Ψ_3, the sum of only the first two or three harmonic functions.

A clever method (too involved to detail here), invented by Schneider[62] for solution of the Dirichlet problem, conjoins use of several theorems of function theory. Applied to the torsion problem, it affords to any desired degree of accuracy the value of ϕ at any point of a cross section which can be mapped piece by piece, overlapping allowed, on a unit circle. Illustratively, Schneider applies his analysis to determination of a set of values of ϕ for a cross section much like that treated by Runge: a cross comprised of five squares, the four outermost sides of the cross being slightly curved. Additionally, he studies the "Zweibogendreick" comprised of the area within the outer perimeters of two intersecting orthogonal circles of equal radii.

In conclusion, we mention Runge[63] has disproved the method advanced by Henneberg[64, 65] for solution of the torsion problem.

APPENDIX

Though none of the following papers are concerned with the torsion problem, the serious student will find them of interest for the reasons noted: by Kryloff[66] and by Kantorovič[67] on Ritz's method—the former contains many references to writings, mostly by Russian authors, not mentioned in the bibliography of reference 15; by Williams,[68] Bickley,[69] and Frazer, Jones, and Skan[70] on Galerkin's method—incidentally, also on the methods of least squares and collocation; by Courant[71] (see also part 4 of reference 15) on the method of gradients—a variational method proclaimed as of considerable power, though as yet it has not (?) been employed for effecting actual numerical solutions; by Hencky[72] and by Goldsbrough[13] on aspects of a certain series method linked with Ritz's method.

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Distribution of Light Intensity within a Scattering Medium

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An integral-differential equation is developed which gives the intensity within a scattering medium as a function of position and direction. The scattering properties of an infinitesimal volume of the medium are assumed known, but no special assumption is made concerning their nature. The equation is put in a form suitable for numerical solution.

INTRODUCTION

THE properties of the light scattered by a plane-parallel layer of a scattering medium have been used to indicate many of the properties of the medium. For example, Hertel and Zervigon[1] have measured the number of fibers in a sample of cotton fibers which have a certain length by a method which essentially depends upon measuring the light which is transmitted by a layer of the sample. Sauer[2] has used the intensity of the light transmitted by a layer of a turbid medium to measure the turbidity of the layer. Also, many workers have used measurements of scattering to indicate the size of particles suspended in some other medium.[3-5]

Some of the methods have depended upon illuminating the entire face of a layer of the medium, and measuring the emerging intensity, while, in other methods, the measurements are made upon the cone into which an incident pencil of parallel rays is spread. Only the former method will be discussed in this paper.

In connection with these methods, a number of workers[6-13] have developed equations which give the light transmitted by a plane-parallel layer of a medium as a function of its thickness, and of the optical properties of the medium. As a result of the complexity of the problem, simplifying assumptions have often been made. One assumption used is that light is scattered only in the direction of the incident beam, or in the reverse direction; another common assumption is that light which is once scattered is not rescattered. The first assumption will be true for certain types of media, e.g., a stack of glass plates; the second might be nearly correct for thin layers, but will not be met for thick layers.

The purpose of the present paper is to obtain an equation which will permit calculation of the variation of light intensity in a layer of a scattering medium, in terms of the scattering properties of an infinitesimal volume of the layer. In principle, the resulting equation can be solved, no matter how complicated these properties may be, although in practice the calculations will be very tedious.

The equation obtained is similar to an equation given in reference 11, but it is felt that in many cases the desired information can be obtained more easily from the equation given here.

I. PROPERTIES OF AN INFINITESIMAL VOLUME OF THE MEDIUM

Consider an infinitesimal volume of the medium situated at the origin of a system of spherical coordinates (see Fig. 1), with a thickness dz and a cross-sectional area dA. Allow light of intensity I to come from the direction of the negative z axis and strike this volume. Then the energy (per second) incident upon the volume will be IdA. Some of this energy will now suffer

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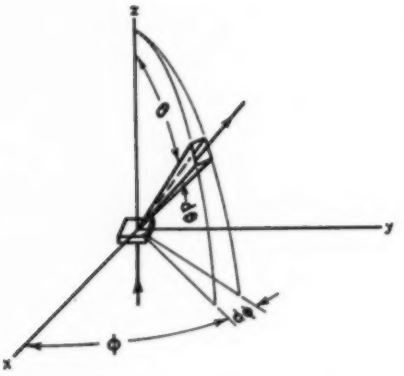

FIG. 1. Angular relations used in defining $\alpha(\theta, \phi)$.

"scattering," using the term to indicate any process by which the direction of the light is altered.

Assuming that the amount of energy scattered is proportional to the thickness of the infinitesimal volume, and depends upon the direction which the scattered light takes, some fraction $\alpha(\theta, \phi)d\omega dz$ of the incident energy will be scattered into the small solid angle $d\omega$ lying about the direction defined by (θ, ϕ). The energy (per second) thus scattered into $d\omega$ will be

$$IdA \cdot \alpha(\theta, \phi)d\omega dz = I\alpha(\theta, \phi)d\omega dV,$$

which depends only upon dV, and not upon the shape of the infinitesimal volume. The total energy scattered by dV will be $\kappa I dV$, where κ is defined as:

$$\kappa = \int_0^{2\pi}\int_0^{\pi} \alpha(\theta, \phi) \sin\theta d\theta d\phi.$$

Two cases are common in practice: If the medium consists of small particles oriented at random, or spherical, the scattering will be symmetrical about the direction of incidence. If the medium consists of cylinders with axes parallel, the scattered light will lie in a plane perpendicular to the axes of the cylinders. Considering only the first case, $\alpha(\theta, \phi)$ must be independent of ϕ, and integrating once:

$$\kappa = 2\pi\int_0^{\pi} \alpha(\theta) \sin\theta d\theta.$$

It will be convenient later to use $\cos\theta$ as the variable, so denoting $2\pi\alpha(\theta, \phi)$ by $\sigma(\cos\theta)$:

$$\kappa = \int_0^{\pi} \sigma(\cos\theta) \sin\theta d\theta. \qquad (1)$$

In addition to scattering energy, the infinitesimal volume may also absorb energy of amount $\beta I dV$.

In certain cases, such as for raindrops, or for particles whose dimensions are comparable with the wave-lengths of the light used, the function $\sigma(\cos\theta)$ could be calculated from theoretical considerations. In general, it will be easier to measure it by taking a thin layer of the material, illuminating it by a narrow pencil of rays, and measuring the intensity scattered in different directions. That scattered directly forward or backward could not be measured, but could be obtained by interpolation.

II. DERIVATION OF THE EQUATION

Figure 2 represents a cross section of a plane-parallel layer of a scattering medium, of thickness d, and of infinite extent in the horizontal direction and in the direction perpendicular to the figure. In case the scattering particles are parallel cylinders, the cross section is to be taken perpendicular to their axes, and the extent of the sample along the direction of the axes is immaterial. There are cases intermediate between this case and that in which the scattering medium contains randomly oriented particles, but these will not be considered.

The sample is considered to be illuminated from above, over the entire face of the sample, by collimated light striking at normal incidence. It will be assumed that the intensity of light at any point within the layer is proportional to the incident intensity. Hence, for simplicity, the incident intensity is taken to be unity.

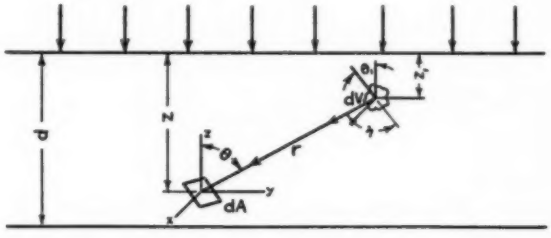

FIG. 2. Cross section of a scattering layer, showing the coordinates used.

In any actual experiment, neither the sample nor the incident beam of light will have the infinite extent assumed above. A criterion for determining the lateral extent required will be given in the last section.

The first case considered will be that in which the scattering particles are oriented at random. Referring again to Fig. 2, dA represents an infinitesimal area at a distance z below the illuminated surface. In dA lies the origin of a system of spherical coordinates r, θ, ϕ. Further, dA is so oriented that its normal has the direction defined by the angles θ and ϕ.

Light which reaches a depth z consists of two parts: an intensity $e^{-(\kappa+\beta)z}$ (per unit incident intensity) which has been transmitted directly, travelling in the forward direction, and an intensity which has reached the point in question after being scattered one or more times. If dV is an infinitesimal volume at a depth z_1 below the top face, some of the energy reaching dA has been scattered from dV.

Denote the energy of scattered light which strikes the area dA, and which is travelling within a small solid angle $d\Omega$ drawn about the (θ, ϕ) direction, by $f(z, \theta, \phi)dAd\Omega$. Then $f(z, \theta, \phi)$ denotes the energy of scattered light per unit area per unit solid angle which exists at a distance z below the upper face, travelling toward the origin along the direction defined by (θ, ϕ).

The same function at the location of dV will be denoted by $f(z_1, \theta_1, \phi_1)$. Then the energy which strikes dV with directions lying within the small solid angle $\sin\theta_1 d\theta_1 d\phi_1$ is $f(z_1, \theta_1, \phi_1) \times \sin\theta_1 d\theta_1 d\phi_1 dA_1$, where dA_1 is the effective area presented to light travelling in this direction. By the remarks of the previous section, a fraction $(1/2\pi)\sigma(\cos\eta)d\omega dt_1$ of this energy will be scattered into directions such as to pass through dA. Here $d\omega = dA/r^2$, dt_1 is the effective thickness of dV (so that $dV = dA_1 dt_1$), and η is the angle between the directions defined by (θ, ϕ) and by (θ_1, ϕ_1), so that

$$\cos\eta = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi-\phi_1).$$

The energy scattered from the small solid angle $\sin\theta_1 d\theta_1 d\phi_1$ into $d\omega$ is then $(1/2\pi)\sigma(\cos\eta) \times f(z_1, \theta_1, \phi_1)\sin\theta_1 d\theta_1 d\phi_1 dV d\omega$. Upon integrating this over θ_1 and ϕ_1, there is obtained the total energy which is rescattered by dV into the proper direction to pass through dA.

In addition, the intensity $\exp[-(\kappa+\beta)z_1]$ reaches dV without being previously scattered; of this, an amount

$$\exp[-(\kappa+\beta)z_1][\sigma(\cos\theta)/2\pi]dVd\omega$$

is scattered so as to pass through dA. Then the total energy scattered by dV into the solid angle $d\omega$ is:

$$(1/2\pi)\Big[\int_0^{2\pi}\int_0^{\pi}\sigma(\cos\eta)f(z_1, \theta_1, \phi_1)\sin\theta_1 d\theta_1 d\phi_1 + \exp(-\gamma z_1)\sigma(\cos\theta)\Big]dVd\omega,$$

writing γ for $(\kappa+\beta)$. Of this energy, a fraction $e^{-\gamma r}$ will arrive at dA; the rest will be absorbed or scattered out of the path. That which reaches dA will be travelling in directions lying within $d\Omega$, the solid angle subtended by dV at dA. This energy gives a contribution to $f(z, \theta, \phi)dAd\Omega$, which we call $df(z, \theta, \phi)dAd\Omega$. Taking dV to be a pillbox with axis parallel to r, of thickness dr, then $dV = r^2 d\Omega dr$, while $dA = r^2 d\omega$. Making all of the indicated substitutions, we obtain the equation:

$$df(z, \theta, \phi)r^2 d\Omega d\omega = (1/2\pi)\Big[\int_0^{2\pi}\int_0^{\pi}\sigma(\cos\eta)f(z_1, \theta_1, \phi_1)\sin\theta_1 d\theta_1 d\phi_1 + \exp(-\gamma z_1)\sigma(\cos\theta)\Big]e^{-\gamma r}r^2 d\Omega dr d\omega. \quad (2)$$

This can be divided through by $r^2 d\Omega d\omega$, and integrated over r from zero to the boundary of the layer. Further, the function $f(z, \theta, \phi)$ must be independent of ϕ. This gives the integral equation for the function $f(z, \theta)$:

$$f(z, \theta) = (1/2\pi)\int\Big[\int_0^{2\pi}\int_0^{\pi}\sigma(\cos\eta)f(z_1, \theta_1) \times \sin\theta_1 d\theta_1 d\phi_1 + \exp(-\gamma z_1)\sigma(\cos\theta)\Big]e^{-\gamma r}dr. \quad (3)$$

The various ranges for r are: for $0 \leqslant \theta < \pi/2$, $0 \leqslant r \leqslant z/\cos\theta$; for $\theta = \pi/2$, $0 \leqslant r \leqslant \infty$; for $\pi/2 < \theta \leqslant \pi$, $0 \leqslant r \leqslant (d-z)/\cos\theta$.

Changing the variable to z_1 by means of the relation $r=(z-z_1)/\cos\theta$:

$$f(z,\theta)=(\exp[-\gamma z/\cos\theta]/2\pi\cos\theta)\times\int^{z}\Big\{\int_0^{2\pi}\int_0^{\pi}\sigma(\cos\eta)f(z_1,\theta_1)\times\exp[(\gamma z_1/\cos\theta)]\sin\theta_1 d\theta_1 d\phi_1+\sigma(\cos\theta)\exp[\gamma z_1(1-\cos\theta)/\cos\theta]\Big\}dz_1 \quad (4)$$

for $\theta\neq\pi/2$, while for $\theta=\pi/2$:

$$f(z,\pi/2)=(1/2\pi\gamma)\Big[\int_0^{2\pi}\int_0^{\pi}\sigma(\cos\eta)f(z,\theta_1)\times\sin\theta_1 d\theta_1 d\phi_1+\sigma(0)e^{-\gamma z}\Big]. \quad (5)$$

The lower limit of integration in the first equation is zero, if $\theta<\pi/2$, or d, if $\theta>\pi/2$.

By differentiating Eq. (4) with respect to z, the result can be put in a simpler form, and one which also holds for $\theta=\pi/2$, so that the troublesome double form is eliminated. Performing the differentiation and solving for $f(z,\theta)$:

$$f(z,\theta)=-(\cos\theta/\gamma)[\partial f(z,\theta)/\partial z]+[\sigma(\cos\theta)/2\pi\gamma]e^{-\gamma z}+(1/2\pi\gamma)\int_0^{2\pi}\int_0^{\pi}\sigma(\cos\eta)f(z,\theta_1)\sin\theta_1 d\theta_1 d\phi_1. \quad (6)$$

The problem for the case in which the scattering medium consists of small cylinders with axes parallel is handled in much the same manner, and only the result will be given. The problem is only two-dimensional, so that $\sigma(\cos\theta)d\theta dz$ is the fraction of the energy striking it which a small thickness dz of the medium scatters into a wedge with dihedral angle $d\theta$. The figure corresponding to Fig. 2 is a cross section of the medium taken perpendicular to the axes of the cylinders, so that all directions in which energy is travelling lie in this plane. If

$$\kappa=\int_{-\pi}^{\pi}\sigma(\cos\theta)d\theta$$

denotes the scattering coefficient, and β the absorption coefficient, the equations corresponding to Eqs. (3) and (6), respectively, are:

$$f(z,\theta)=\int\Big[\int_{-\pi}^{\pi}\sigma(\cos\eta)f(z_1,\theta_1)d\theta_1+\exp(-\gamma z_1)\sigma(\cos\theta)\Big]e^{-\gamma r}dr, \quad (7)$$

$$f(z,\theta)=-(\cos\theta/\gamma)[\partial f(z,\theta)/\partial z]+[\sigma(\cos\theta)/\gamma]e^{-\gamma z}+(1/\gamma)\int_{-\pi}^{\pi}\sigma(\cos\eta)f(z,\theta_1)d\theta_1, \quad (8)$$

where $\cos\eta=\cos(\theta-\theta_1)$; the definitions of other quantities, and the ranges of integration for r, are as before.

III. USE OF "FORWARD" AND "BACKWARD" FLUX

Several authors[7–10] have treated the problem by assuming that all of the scattered energy travelling in directions for which θ lies between 0 and $\pi/2$ is equivalent to a forward flux F_1, travelling in the direction of the incident light, and that the energy with directions for which θ is greater than $\pi/2$ is equivalent to a backward flux F_2. Then light is assumed to be scattered either forward with a coefficient $\zeta\kappa$, or backward with a coefficient $(1-\zeta)\kappa$. Ryde and Cooper[10] have elaborated this treatment further by using different coefficients ζ and ζ' for the parallel, directly transmitted, light, and for the scattered, diffused, light. In deriving the coefficient for the diffused light, however, they assumed the diffused light to be distributed equally in direction, whereas the coefficient would actually have to depend upon the angular distribution of the light, and hence upon position within the scattering layer.

Equation (8) can be reduced to the form given by Silberstein,[7] if $\sigma(\cos\theta)$ and $f(z,\theta)$ have the following forms:

For $-\pi/2\leqslant\theta\leqslant\pi/2$, $\sigma(\cos\theta)=\zeta\kappa\delta(\theta)$, $f(z,\theta)=F_1\delta(\theta)$.

For $\pi/2\leqslant\theta\leqslant 3\pi/2$, $\sigma(\cos\theta)=(1-\zeta)\kappa\delta(\theta-\pi)$, $f(z,\theta)=F_2\delta(\theta-\pi)$.

For convenience in making these definitions, the range of θ has been changed from $(-\pi, \pi)$ to $(-\pi/2, 3\pi/2)$. $\delta(\theta)$ is the delta function,[14] which has the properties:

$$\delta(x-a)=0, \quad x\neq a,$$

and

$$\int f(x)\delta(x-a)dx=f(a),$$

provided that the range of integration includes the point a. Substituting the values of σ and $f(z, \theta)$, we obtain, for $|\theta| \leqslant \pi/2$:

$$\gamma F_1\delta(\theta)=-\cos\theta F_1'\delta(\theta)+\zeta\kappa e^{-\gamma z}\delta(\theta)$$
$$+\int_{-\pi/2}^{\pi/2}\zeta\kappa\delta(\theta-\theta_1)F_1\delta(\theta_1)d\theta_1$$
$$+\int_{\pi/2}^{3\pi/2}(1-\zeta)\kappa\delta(\theta-\theta_1-\pi)F_2\delta(\theta_1-\pi)d\theta_1.$$

Evaluating the integrals:

$$\gamma F_1\delta(\theta)=-\cos\theta F_1'\delta(\theta)+\zeta\kappa e^{-\gamma z}\delta(\theta)$$
$$+\zeta\kappa F_1\delta(\theta)+(1-\zeta)\kappa F_2\delta(\theta).$$

Integrating over θ from $-\pi/2$ to $\pi/2$,

$$\gamma F_1=-F_1'+\zeta\kappa e^{-\gamma z}+\zeta\kappa F_1+(1-\zeta)\kappa F_2. \quad (9a)$$

Proceeding similarly for the case $\pi/2\leqslant\theta\leqslant 3\pi/2$,

$$\gamma F_2=F_2'+(1-\zeta)\kappa e^{-\gamma z}+(1-\zeta)\kappa F_1+\zeta\kappa F_2. \quad (9b)$$

If Eqs. (9a) and (9b) are solved for the derivatives F_1' and F_2', they are seen to be identical with the differential equations given by Silberstein, of which the integration is given in reference 7.

IV. DISCUSSION

The appearance of the term $(\sigma(\cos\theta)/2\pi\gamma)e^{-\gamma z}$ in Eq. (6) is interesting, for if σ were a constant equal to $\kappa/2$, this term becomes $(1/4\pi)e^{-\kappa z}$, neglecting the absorption coefficient compared to the scattering coefficient. This represents a total intensity at any point of $e^{-\kappa z}$, distributed evenly in direction; and represents a first approximation to the solution of the problem. Similar remarks apply to Eq. (8).

[14] For a discussion of this function see P. A. M. Dirac, *Quantum Mechanics* (University Press, Oxford, 1935), second edition, Chapter IV.

The case for which σ equals $\kappa/2$, and there is no absorption, so that $\gamma=\kappa$, is interesting from another consideration. Upon making these substitutions in Eq. (6), it can readily be seen that the equation contains only the combination κz; that is, κz is a dimensionless variable, and κd is a dimensionless parameter of the system which determines how much light is transmitted by the medium without suffering scattering. Under other conditions, γ always appears as γz, or as (σ/γ). If the total number of particles in a scattering layer remains unchanged, these quantities remain unchanged if the layer is compressed or expanded. In this sense, γd is a parameter of the system, with the same meaning as κd has above.

The form of Eqs. (6) and (8) is such that they will probably defy analytic solution, even for very simple forms of the function $\sigma(\cos\theta)$. It will then be necessary to obtain the function $f(z, \theta)$ by methods of successive numerical approximation. If absorption can be neglected, the equations are somewhat simplified; and if, in addition, the scattering can be assumed spherically symmetrical, the resulting equations are considerably simpler. The integral

$$\int_0^{2\pi}\sigma(\cos\eta)d\phi_1$$

can be evaluated as a function of θ and θ_1 as soon as the form of σ is known, so that only one integration need be performed for each step in the approximation process. In cases where it is felt that the derivative $(\delta f/\delta z)$ cannot be obtained numerically with sufficient accuracy, the double integral form of Eq. (4) can be used.

From Eq. (3), an approximate criterion for the lateral extent required of the sample can be developed. Referring to Eq. (3), the quantity in square brackets may be considered to behave roughly like $\exp(-\gamma z_1)$. Then, if a designates the horizontal distance from the origin of r to the edge of the layer, the error in calculating $f(z, \theta)$ will be less than one percent if:

$$e^{-\gamma z}\int_0^{z/\cos\theta}e^{-\gamma r(1-\cos\theta)}dr$$
$$\geqslant 100\cdot e^{-\gamma z}\int_{a/\sin\theta}^{z/\cos\theta}e^{-\gamma r(1-\cos\theta)}dr,$$

since $z_1+r=z+r(1-\cos\theta)$. That is, the contribution to the integral from points beyond the actual edge of the layer is less than one percent of the entire integral. Evaluating the integrals, it may be seen that the largest value of a required is for $\theta=90°$. Substituting this value of θ, the above condition becomes:

$$e^{-\gamma a}\leqslant 0.01.$$

The necessary lateral extent is then independent of the thickness of the sample. Of course, actually cutting away the portion of the sample lying at a greater distance than a from the point where measurements are being made does not produce the same effect as ignoring that part in the calculations. In spite of this, the criterion given should prove useful.

There are several uses for the function $f(z, \theta)$ which suggest themselves. If one is interested in the directed intensity at any point, that is, of course, just $f(z, \theta)$ itself. The emerging intensity in any direction is the value of $f(z, \theta)$ at $z=d$. To obtain the total intensity received upon a flat surface, it will be necessary to multiply $f(z, \theta)$ by the cosine of the appropriate angle, and to integrate over all directions from which light is received. This process, if carried out for a surface coincident with the lower face of the sample, and $e^{-\gamma d}$ added to the result, will give the total transparency of the layer.

The author wishes to express his appreciation to Dr. K. L. Hertel, of this department, who suggested the problem, and to Dr. W. G. Pollard, who has contributed many valuable suggestions.

Applications of the Operational Calculus to the Theory of Structures

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The differential equations governing the deflection of structural elements such as loaded cords and beams freely suspended or resting on an elastic foundation are solved operationally. The operational method simplifies the analysis because it reduces the number of arbitrary constants to be determined and enables the effect of concentrated loads or applied couples to be determined directly.

INTRODUCTION

THE determination of the equilibrium configurations of loaded cords and beams leads to the solution of ordinary differential equations subject to multi-point boundary conditions. In the usual applications, however, the boundary conditions are specified at two points. Many analytical and graphical methods[1] have been devised to solve problems of this nature. In some instances, purely analytical procedures are very cumbersome. It is the purpose of this discussion to illustrate how problems of this type may be solved simply and directly by the use of the Laplace transform or operational method. It may be mentioned that in the usual applications of this method, the differential equations under consideration are solved subject to one-point boundary conditions; however, the utility of this method in certain multi-point boundary value problems will be apparent from this discussion.

I. THE LAPLACIAN TRANSFORMATION

a. General Considerations

Before proceeding, it may be well to review briefly the theory on which the modern operational calculus is based.[2] The modern theory of

[1] S. Timoshenko, *Strength of Materials* (D. Van Nostrand Company, New York, 1932).

[2] L. A. Pipes, J. App. Phys. **10** (1939).

the operational calculus, a mathematical technique which has proved to be such a powerful mathematical tool in the solution of the linear differential equations of applied mathematics is based on the Laplacian transformation. This transformation is usually written in the following form:

$$g(p)=p\int_0^\infty \epsilon^{-px}h(x)dx \quad Re\ p>0. \tag{1}$$

In the usual applications of the theory, $h(x)$ is a function for which the definite integral (1) exists and satisfies the condition,

$$h(x)=0 \quad \text{for } x<0. \tag{2}$$

A rigorous discussion of the restrictions that must be imposed on a function $h(x)$ in order that it may have a Laplacian transformation is given elsewhere.[2-4] It may be said, however, that the functions encountered in the various physical problems where the transform method is applicable satisfy the required conditions. In most of the modern literature on operational or Laplacian transform methods, the functional relation expressed between $g(p)$ and $h(x)$ is written in the following form:

$$g(p)=Lh(x). \tag{3}$$

The L denotes the "Laplacian transform of" and greatly shortens the writing. The relation between $h(x)$ and $g(p)$ is also written in the form

$$h(x)=L^{-1}g(p). \tag{4}$$

In such a case we say that $h(x)$ is the inverse Laplacian transform of $h(x)$.

A table of definite integrals of the form (1) may be used to obtain many transforms. A collection of these transforms useful in applications to the theory of structures will be found in the appendix.

b. The Laplacian Transforms of Derivatives

In the solution of differential equations by the Laplacian transformation, it is necessary to transform the derivatives of a function. It is easy to show by integrating by parts[2] that the following relations exist:

If

$$Ly(x)=Y(p), \tag{5}$$

then

$$L(dy/dx)=pY-py_0, \tag{6}$$

$$L(d^2y/dx^2)=p^2Y-p^2y_0-py_1, \tag{7}$$

$$L(d^3y/dx^3)=p^3Y-p^3y_0-p^2y_1-py_2, \tag{8}$$

$$L(d^4y/dx^4)=p^4Y-p^4y_0-p^3y_1-p^2y_2-py_3, \tag{9}$$

where

$$y_r=(d^ry/dx^r) \text{ evaluated at } x=0. \tag{10}$$

These relations are of great importance in the solution of differential equations by the use of the Laplacian transformation.

II. DEFLECTION OF LOADED FLEXIBLE CORDS

Let us consider the deflection of the tightly stretched cord of Fig. 1.

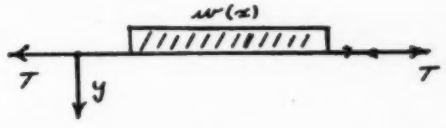

FIG. 1.

It may be shown[5] that the equation governing the small deflection y of a cord tightly stretched by a tension T and loaded by a load of $w(x)$ per unit length is

$$d^2y/dx^2=-w(x)/T, \tag{11}$$

where y is measured downward and $w(x)$ is positive when acting in a downward direction.

To solve this equation, let us introduce

$$Ly(x)=Y(p), \tag{12}$$

$$Lw(x)=g(p). \tag{13}$$

Then by using (7), Eq. (11) is transformed to

$$p^2Y-p^2y_0-py_1=-g(p)/T. \tag{14}$$

1. Uniform Load

To illustrate the method, let us first consider the case in which

$$w(x)=w_0. \tag{15}$$

[2] N. W. McLachlan, *Complex Variable and Operational Calculus* (Cambridge, 1939).
[4] M. F. Gardner and J. L. Barnes, *Transients in Linear Systems* (Wiley, 1942).
[5] T. Von Kármán and M. A. Biot, *Mathematical Methods in Engineering* (McGraw-Hill, 1940), Chapter 7.

In this case (see Appendix), we have

$$Lw_0 = w_0. \tag{16}$$

If the two ends of the span are at equal heights, then

$$y=0 \text{ at } x=0, \quad y=0 \text{ at } x=s. \tag{17}$$

Hence

$$y_0 = 0. \tag{18}$$

Equation (14) then becomes

$$Y = (y_1/P) - (w_0/T)(1/p^2). \tag{19}$$

By the table of transforms in the appendix, the inverse transform of (15) is

$$y = y_1 x - (w_0/T)(x^2/2). \tag{20}$$

To determine y_1 we make use of the boundary Eq. (17) and obtain

$$y_1 = (w_0/T)(s/2). \tag{21}$$

Substituting this into (16) we obtain

$$y = (w_0/2T)(sx - x^2) \tag{22}$$

for the deflection of the cord.

2. Uniform Load Extending over Part of Span

Let us consider the case of loading shown in Fig. 2. In this case, the transform of the load is

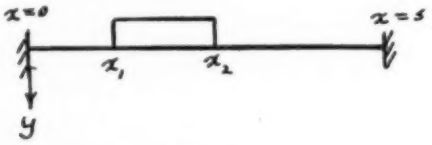

FIG. 2.

given by (see Appendix)

$$g(p) = w_0(\epsilon^{-x_1 p} - \epsilon^{-x_2 p}). \tag{23}$$

Equation (14) becomes

$$Y = \frac{y_1}{p} - \frac{w_0}{T}\left(\frac{\epsilon^{-x_1 p} - \epsilon^{-x_2 p}}{p^2}\right). \tag{24}$$

The inverse transform of this is (see Appendix)

$$\begin{aligned} &y = y_1 x \quad 0<x<x_1, \\ &y = y_1 x - (w_0/2T)(x-x_1)^2 \quad x_1<x<x_2, \\ &y = y_1 x - (w_0/2T)(x-x_1)^2 \\ &\qquad + (w_0/2T)(x-x_2)^2 \quad x_2<x<s. \end{aligned} \tag{25}$$

To determine y_1 we make use of the boundary condition

$$y=0 \quad \text{at} \quad x=s. \tag{26}$$

Inserting this in the third equation of (25) we obtain

$$y_1 = (w_0/2Ts)[(s-x_1)^2 - (s-x_2)^2]. \tag{27}$$

The deflection curve is given by (25) with this value of y_1.

3. Concentrated Load

Let us compute the deflection of a flexible cord fixed at both ends and supporting a concentrafed load W at a distance of x from the left end as shown in Fig. 3.

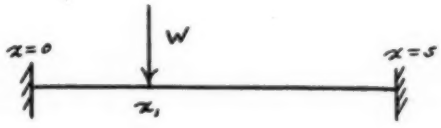

FIG. 3.

To determine this deflection, we may use Eq. (14). It is shown in the appendix that the transform of a concentrated load W situated at $x=x_1$ is given by

$$g(p) = W\epsilon^{-x_1 p}. \tag{28}$$

Substituting this into (14), we obtain

$$Y = \frac{1}{p} y_1 - \frac{W}{T}\epsilon^{-x_1 p}\frac{1}{p}. \tag{29}$$

Computing the inverse transform by using the table in the appendix, we obtain,

$$\begin{aligned} &y = y_1 x \quad 0<x<x_1, \\ &y = y_1 x - (W/T)(x-x_1) \quad x_1<x<s. \end{aligned} \tag{30}$$

The constant y_1 may be determined from (30) by realizing that

$$y=0 \quad \text{at} \quad x=s. \tag{31}$$

Hence we obtain

$$y_1 = (W/Ts)(s-x_1). \tag{32}$$

Substituting this value of y_1 into (30) we obtain the following function for the deflection.

$$\begin{aligned} &y = (W/T)x[1-(x_1/s)] \quad 0<x<x_1, \\ &y = (W/T)x_1[1-(x/s)] \quad x_1<x<s. \end{aligned} \tag{33}$$

4. Arbitrary Load

To determine the deflection produced by an arbitrary load $w(x)$ per unit length, $w(x)$ we have from Eq. (14)

$$Y=\frac{1}{p}y_1-\frac{1}{T}\frac{g(p)}{p^2}. \tag{34}$$

To compute the inverse transform of $g(p)/p^2$ we use the Faltung theorem (see Appendix); we thus obtain

$$L^{-1}\frac{g(p)}{p^2}=\int_0^x w(u)(x-u)du. \tag{35}$$

The inverse transform of (34) is

$$y=y_1x-\frac{1}{T}\int_0^x w(u)(x-u)du. \tag{36}$$

The condition that y must vanish at $x=s$ enables y_1 to be determined. Its value is

$$y_1=\frac{1}{Ts}\int_0^s w(u)(s-u)du. \tag{37}$$

Substituting this value of y_1 into (36) we obtain

$$y=\frac{x}{Ts}\int_0^s w(u)(s-u)du-\frac{1}{T}\int_0^x w(u)(x-u)du \tag{38}$$

for the deflection of the cord under the influence of a general load $w(x)$.

III. STRETCHED CORD WITH ELASTIC SUPPORT

Let us consider the deflection of a cord stretched by means of a tension T resting on an elastic foundation and under the influence of a certain load $w(x)$ per unit length.

The differential equation governing the deflection of such a string is given by[5]

$$T\frac{d^2y}{dx^2}-ky=-w(x). \tag{39}$$

In this equation, T is the tension of the cord, $w(x)$ is the load per unit length, and k a constant of proportionality which depends on the stiffness of the assumed elastic foundation.

That is, the vertical deflection of the cord is restrained by a large number of springs such that their action can be replaced by a distributed restoring force per unit length equal to ky.

Equation (39) may be written in the form

$$\frac{d^2y}{dx^2}-c^2y=\frac{-w(x)}{T}, \tag{40}$$

where

$$c^2=k/T. \tag{41}$$

To solve this equation operationally, let

$$Ly(x)=Y(p), \tag{42}$$

$$Lw(x)=g(p). \tag{43}$$

Equation (40) is thus transformed to

$$(p^2-c^2)\,Y=p^2y_0+py_1-[g(p)/T]. \tag{44}$$

1. Cord Fixed at Its End, Concentrated Load

Let us consider the case shown in Fig. 4.

FIG. 4.

In this case, we wish to determine the vertical deflection y produced by the application of a concentrated load W at a distance x_1 from the left end support. The cord is under a tension T and rests on an elastic foundation of elastic constant k.

We have the following boundary conditions:

$$y_0=0, \quad y=0 \quad \text{at} \quad x=s. \tag{45}$$

The transform of the concentrated load at $x=x_1$ is

$$g(p)=\epsilon^{-x_1p}pW. \tag{46}$$

Hence Eq. (44) becomes

$$Y=\frac{py_1}{(p^2-c^2)}-\frac{W}{T}\epsilon^{-x_1p}\frac{p}{(p^2-c^2)}. \tag{47}$$

Consulting the table of transforms in the appendix, we find the inverse transform to be

$$y=y_1\frac{\sinh(cx)}{c} \quad 0<x<x_1, \tag{48}$$

$$y=y_1\frac{\sinh(cx)}{c}-\frac{W}{Tc}\sinh[c(x-x_1)] \quad x_1<x<s. \tag{49}$$

If we use the boundary condition, that the deflection must vanish at $x=s$, we obtain

$$y_1=\frac{W}{T}\frac{\sinh[c(s-x_1)]}{\sinh(cs)}. \tag{50}$$

Substituting this value of y_1 in Eqs. (48) and (49), we obtain the following expression for the deflection of the cord.

$$\begin{aligned} y&=\frac{W}{Tc}\cdot\frac{\sinh[c(s-x_1)]\sinh(cx)}{\sinh(cs)}, \quad 0<x<x_1,\\ y&=\frac{W}{Tc}\cdot\frac{\sinh[c(s-x_1)]}{\sinh(cs)}\sinh(cx)\\ &\quad -\frac{W}{Tc}\cdot\sinh[c(x-x_1)] \quad x_1<x<s. \end{aligned} \tag{51}$$

2. Infinite Cord Elastically Supported, Concentrated Load

The deflection of an infinitely long elastically supported cord under the influence of a concentrated load W may be obtained as a special case of (41).

In order to obtain the deflection in this case, let

$$z=(x-x_1), \tag{52}$$

$$x_1=s/2. \tag{53}$$

Making these substitutions in (51), we obtain

$$y=\frac{W}{Tc}\sinh\left(\frac{cs}{2}\right)\sinh\left(\frac{cs}{2}+cz\right)-\frac{W}{Tc}\sinh(cz), \quad z>0. \tag{54}$$

Then

$$\operatorname*{Limit}_{s\to\infty} y=\frac{W}{2Tc}\epsilon^{-cz}=\frac{W}{2(kt)^{\frac{1}{2}}}\epsilon^{-cz} \quad z>0. \tag{55}$$

This is the required deflection and z is measured from the point of application of the load.

3. Elastically Supported Cord with Uniform Load

Let us consider the case shown in Fig. 5.

FIG. 5.

In this case, the cord is loaded with a uniform load w_0 per unit length. The load extends from x_1 to x_2.

In this case, we have

$$Lw(x)=w_0(\epsilon^{-x_1p}-\epsilon^{-x_2p}). \tag{56}$$

Equation (44) becomes

$$Y=y_1\frac{p}{(p^2-c^2)}-\frac{w_0}{T}\left(\frac{\epsilon^{-x_1p}-\epsilon^{-x_2p}}{(p^2-c^2)}\right). \tag{57}$$

Consulting the table of transforms in the appendix, we obtain

$$\begin{aligned} y&=y_1\frac{\sinh(cx)}{c} \quad 0<x<x_1,\\ y&=y_1\frac{\sinh(cx)}{c}-\frac{w_0}{Tc^2}\{\cosh c(x-x_1)-1\} \quad x_1<x<x_2,\\ y&=y_1\frac{\sinh(cx)}{c}-\frac{w_0}{Tc^2}\cosh c(x-x_1)\\ &\quad +\frac{w_0}{Tc^2}\cosh c(x-x_2) \quad x_2<x<s. \end{aligned} \tag{58}$$

Using the boundary condition that the deflection must vanish at $x=s$, we obtain the following value for the constant y_1:

$$y_1=\frac{w_0}{Tc\sinh(cs)}\{\cosh c(s-x_1)-\cosh c(s-x_2)\}. \tag{59}$$

Substituting this value of y_1 into (58) gives the required deflection.

4. Elastically Supported Cord, General Loading

The transform of the deflection of an elastically supported cord under the influence of a general load $w(x)$, is by Eq. (44)

$$Y=\frac{py_1}{(p^2-cp)}-\frac{1}{T}\frac{g(p)}{(p^2-c^2)}, \tag{60}$$

where

$$Lw(x)=g(p). \tag{61}$$

By the Faltung theorem in the appendix we have

$$L^{-1}\frac{g(p)}{(p^2-c^2)}=\frac{1}{c}\int_0^x w(u)\sinh\left[c(x-u)\right]du. \tag{62}$$

The inverse transform of Y is thus given by

$$y=y_1\frac{\sinh(cx)}{c}-\frac{1}{Tc}\int_0^x w(u)\sinh\left[c(x-u)\right]du. \tag{63}$$

The fact that the deflection vanishes at $x=s$ leads to the following value of y_1:

$$y_1=\frac{1}{T\sinh(cs)}\int_0^s w(u)\sinh\left[c(s-u)\right]du. \tag{64}$$

Substituting this value of y_1 into (63) gives the required deflection.

IV. THE DEFLECTION OF BEAMS BY TRANSVERSE FORCES

The deflection y of a uniform beam of flexural rigidity EI satisfies the following differential equations:[1, 5]

$$EI(d^4y/dx^4)=w(x)=\text{load per unit length}. \tag{65}$$

$$F=-EI(d^3y/dx^3)=\text{the shear force}. \tag{66}$$

$$M=-EI(d^2y/dx^2)=\text{the bending moment}. \tag{67}$$

If the load per unit length is known, then the deflection y may be determined by solving Eq. (65) subject to the boundary conditions of the particular problem under consideration. In order to solve (65) operationally, let

$$Ly(x)=Y(p), \tag{68}$$

$$Lw(x)=g(p). \tag{69}$$

Using Eq. (9) for the transform of the fourth derivative, Eq. (65) may be transformed into

$$Y=\frac{1}{EI}\frac{g(p)}{p^4}+y_0+\frac{y_1}{p}+\frac{y_2}{p^2}+\frac{y_3}{p^3}. \tag{70}$$

This equation is basic in the calculation of the deflection of beams.

1. Uniform Beam Clamped Horizontally at Both Ends Under the Influence of a Uniformly Distributed Load

Consider the deflection of the beam shown in Fig. 6.

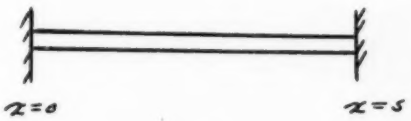

FIG. 6.

In this case, the deflection and slope at $x=0$ are both equal to zero, hence

$$y_0=y_1=0. \tag{71}$$

The transform of the uniform load w_0 is given by

$$g(p)=w_0. \tag{72}$$

Hence Eq. (70) becomes

$$Y=\frac{w_0}{EI}\frac{1}{p^4}+\frac{y_2}{p^2}+\frac{y_3}{p^3} \tag{73}$$

and

$$y=\frac{w_0}{EI}\frac{x^4}{4!}+y_2\frac{x^2}{2!}+y_3\frac{x^3}{3!}. \tag{74}$$

The constants y_2 and y_3 are determined by the condition

$$y=dy/dx=0 \quad \text{at } x=s. \tag{75}$$

These conditions yield

$$y_1=-(1/12)(w_0s/EI). \tag{76}$$

$$y_2=(1/12)(w_0s^2/EI). \tag{77}$$

Substituting these values of y_1 and y_2 into (74) we obtain the following equation for the deflection:

$$y=(1/24)(w_0/EI)x^2(s-x)^2. \tag{78}$$

2. Uniform Beam Clamped Horizontally at Both Ends and Carrying a Concentrated Load

Consider the problem of determining the deflection of the beam shown in Fig. 7.

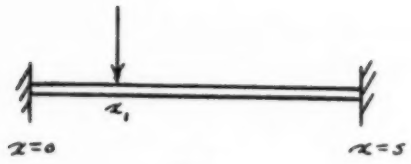

FIG. 7.

In this case, the transform of the concentrated load is

$$g(p)=Wp\epsilon^{-x_1p}. \tag{79}$$

Since the deflection and slope are zero at $x=s$, Eq. (70) for this case becomes

$$Y=\frac{W}{EI}\frac{\epsilon^{-x_1p}}{p^3}+\frac{y_2}{p^2}+\frac{y_3}{p^3}. \tag{80}$$

The inverse transform of Y is

$$\begin{aligned} y&=y_2\frac{x^2}{2!}+y_3\frac{x^3}{3!} \quad 0<x<x_1,\\ y&=\frac{W}{6EI}(x-x_1)^3+y_2\frac{x^2}{2}+y_3\frac{x^3}{6} \quad x_1<x<s. \end{aligned} \tag{81}$$

The conditions that the deflection and slope must vanish at $x=s$ enables y_1 and y_2 to be determined. Inserting these values of y_2 and y_3 into (81) we obtain the following equations for the deflection:

$$\begin{aligned} y&=-\frac{1}{6}\frac{W}{EI}\frac{x^2}{s^3}(s-x_1)^2\{3x_1s-(s+2x_1)x\} \quad 0<x<x_1,\\ y&=-\frac{1}{6}\frac{W}{EI}\frac{x_1^2}{s^3}(s-x)^2\{(3s-2x_1)x-x_1s\} \quad x_1<x<s. \end{aligned} \tag{82}$$

3. Uniform Beam Clamped Horizontally at One End and Free at the Other Carrying a Concentrated Load

As another example of the general method, consider the problem of determining the deflection of the beam shown in Fig. 8.

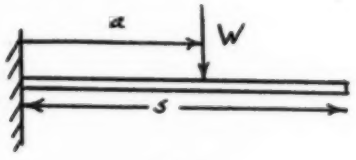

FIG. 8.

In this case, the basic Eq. (70) becomes

$$Y=\frac{W}{EI}\frac{\epsilon^{-ap}}{p^3}+\frac{y_2}{p^2}+\frac{y_3}{p^3}. \tag{83}$$

The inverse transform of Y is

$$\begin{aligned} y&=y_2\frac{x^2}{2}+y_3\frac{x^3}{6} \quad 0<x<a,\\ y&=\frac{W}{6EI}(x-a)^3+y_2\frac{x^2}{2}+y_3\frac{x^3}{6} \quad a<x<s. \end{aligned} \tag{84}$$

In this case, since the end $x=s$ is free, it follows that the bending moment and shear force at that point must vanish. By (66) and (67), we therefore have

$$d^2y/dx^2=d^3y/dx^3=0 \quad \text{at } x=s. \tag{85}$$

Inserting these conditions in (84), we obtain

$$y_2=aW/EI, \tag{86}$$

$$y_3=-W/EI. \tag{87}$$

Substituting these values into (84) we obtain the following equation for the deflection:

$$\begin{aligned} y&=\frac{Wx^2}{EI}\left(\frac{a}{2}-\frac{x}{6}\right) \quad 0<x<a,\\ y&=\frac{Wa^2}{EI}\left(\frac{x}{2}-\frac{a}{6}\right) \quad a<x<s. \end{aligned} \tag{88}$$

4. Uniform Beam Clamped Horizontally at Both Ends Under the Influence of an Applied Couple

Consider the case shown in Fig. 9.

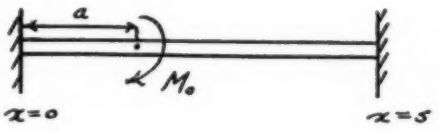

FIG. 9.

It is required to determine the deflection of the beam produced by the application of a couple M_0 at $x=a$. The application of the couple may be regarded as the limit of the loading shown in Fig. 10.

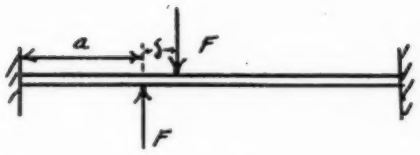

FIG. 10.

That is, the couple may be regarded as the limiting form of loading as shown in Fig. 10 provided that

$$\lim_{\substack{F\to\infty \\ \delta\to 0}} (F\delta) = M_0. \tag{89}$$

To calculate the deflection in this case, we write Eq. (70) in the form:

$$Y = \frac{1}{EI}\frac{g(p)}{p^4} + \frac{y_2}{p^2} + \frac{y_3}{p^3}, \tag{90}$$

where $g(p)$ is the transform of the loading shown in Fig. 10, that is

$$\begin{aligned} g(p) &= -pF\epsilon^{-ap} + pF\epsilon^{-(a+\delta)p} \\ &= pF\epsilon^{-ap}(\epsilon^{-\delta p} - 1). \end{aligned} \tag{91}$$

Now

$$\begin{aligned} \lim_{\substack{\delta\to 0 \\ F\to\infty}} g(p) &= \lim_{\substack{\delta\to 0 \\ F\to\infty}} -(F\delta)p^2\epsilon^{-ap}, \\ &= -M_0 p^2 \epsilon^{-ap}. \end{aligned} \tag{92}$$

This is the transform of a couple of strength M_0 impressed at $x=a$. Substituting this value of $g(p)$ in Eq. (90), we obtain

$$Y = -\frac{M_0}{EI}\frac{\epsilon^{-ap}}{p^2} + \frac{y_2}{p^2} + \frac{y_3}{p^3}. \tag{93}$$

The inverse transform of y_1 is

$$\begin{aligned} y &= y_2\frac{x^2}{2} + \frac{y_3 x^3}{6} \qquad 0<x<a, \\ y &= \frac{-M_0}{EI}\frac{(x-a)^2}{2} + y_2\frac{x^2}{2} + y_3\frac{x^3}{6} \qquad a<x<s. \end{aligned} \tag{94}$$

The condition that the slope and the deflection must vanish at $x=s$ leads to the following values of the constants y_2 and y_3:

$$y_2 = \frac{M_0}{s^2 EI}\{s^2 - 4as + 3a^2\}, \tag{95}$$

$$y_3 = \frac{6M_0 a(s-a)}{s^3 EI}. \tag{96}$$

Substituting these values of y_2 and y_3 in Eq. (94) gives the required deflection.

V. DEFLECTION OF BEAMS ON ELASTIC FOUNDATION

Let us consider the deflection of a uniform beam that is attached to a fixed base by means of an elastic medium. The deflection y of a beam of flexural rigidity EI resting on such a foundation satisfies the differential equation[1, 5]

$$EI(d^4y/dx^4) + ky = w(x). \tag{97}$$

The factor k is called the modulus of the foundation and depends on the elastic property of the foundation. $w(x)$ is the load per unit length applied to the beam. For simplicity, let us write

$$a = (k/4EI)^{\frac{1}{4}}. \tag{98}$$

Equation (97) may then be written in the form

$$(d^4y/dx^4) + 4a^4y = w(x)/EI. \tag{99}$$

To solve this equation operationally, we write

$$Y = Ly(x), \tag{100}$$

$$g(p) = Lw(x). \tag{101}$$

Equation (99) is then transformed to

$$\begin{aligned} (p^4+4a^4)\,Y = [g(p)/EI] &+ p^4y_0 \\ &+ p^3y_1 + p^2y_2 + py_3. \end{aligned} \tag{102}$$

This equation is basic in the computation of the deflection of loaded beams on elastic foundations.

1. Beam on Elastic Foundation Clamped Horizontally at Both Ends Under the Influence of a Concentrated Load

Let us consider the deflection of the beam shown in Fig. 11.

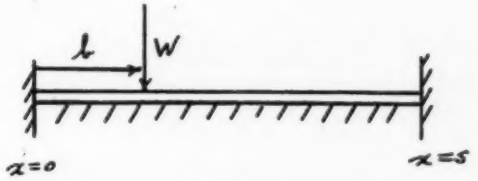

FIG. 11.

In this case, the conditions that the slope and

deflection at $x=0$ must vanish lead to

$$y_0=y_1=0. \tag{103}$$

The transform of the concentrated load W, acting at $x=b$, is

$$g(p)=Wp\epsilon^{-pb}. \tag{104}$$

Hence in this case, Eq. (102) becomes

$$Y=\frac{W}{EI}\frac{p\epsilon^{-pb}}{(p^4+4a^4)}+y_2\frac{p^2}{(p^4+4a^4)}+y_3\frac{p}{(p^4+4a^4)} \tag{105}$$
$$=Ly.$$

The transform of dy/dx is given by

$$L\left(\frac{dy}{dx}\right)=\frac{Wp^2\epsilon^{-pb}}{EI(p^4+4a^4)}+y_2\frac{p^3}{(p^4+4a^4)}+y_3\frac{p^2}{(p^4+4a^4)}. \tag{106}$$

Now let

$$L^{-1}\frac{p}{(p^4+4a^4)}=\phi_1(x), \tag{107}$$

$$L^{-1}\frac{p^2}{(p^4+4a^4)}=\phi_2(x), \tag{108}$$

$$L^{-1}\frac{p^3}{(p^4+4a^4)}=\phi_3(x), \tag{109}$$

$$L^{-1}\frac{p^4}{(p^4+4a^4)}=\phi_4(x). \tag{110}$$

The values of $\phi_r(x)$ are found in the Appendix. In terms of these functions, we have the inverse of Y given by (105) in the form

$$y=y_2\phi_2(x)+y_3\phi_1(x) \quad 0<x<b, \tag{111}$$
$$y=(W/EI)\phi_1(x-b)+y_2\phi_2(x)+y_3\phi_1(x) \quad b<x<s.$$

The constants y_2 and y_3 may be found from the condition that the deflection and the slope must vanish at $x=s$. From (111) we thus obtain

$$0=(W/EI)\phi_1(s-b)+y_2\phi_2(s)+y_3\phi_1(s). \tag{112}$$

And from the inverse of (106) we have

$$0=(W/EI)\phi_2(s-b)+y_2\phi_3(s)+y_3\phi_2(s). \tag{113}$$

These two equations may be solved for the constants y_2 and y_3. This gives

$$y_2=\frac{W}{EI}\frac{[\phi_1(s)\phi_2(s-b)-\phi_2(s)\phi_1(s-b)]}{[\phi_2^2(s)-\phi_3(s)\phi_1(s)]}, \tag{114}$$

$$y_3=\frac{W}{E_I}\frac{[\phi_2(s)\phi_2(s-b)-\phi_3(s)\phi_1(s-b)]}{[\phi_1(s)\phi_3(s)-\phi_2^2(s)]}. \tag{115}$$

The deflection is obtained by substituting these values of y_2 and y_3 into (111).

2. Beam on Elastic Foundation, Clamped Horizontally at Both Ends Under the Influence of a General Load

In this case, since the deflection and slope vanish at $x=0$, Eq. (102) becomes

$$Y=\frac{1}{EI}\frac{g(p)}{(p^4+4a^4)}+\frac{p^2y_2}{(p^4+4a^4)}+\frac{py_3}{(p^4+4a^4)}=Ly. \tag{116}$$

The inverse transform of the slope is given by

$$L\left(\frac{dy}{dx}\right)=\frac{1}{EI}\frac{pg(p)}{(p^4+4a^4)}+\frac{p^3y_2}{(p^4+4a^4)}+\frac{p^2y_3}{(p^4+4a^4)}. \tag{117}$$

By the Faltung theorem, we have

$$L^{-1}\frac{g(p)}{(p^4+4a^4)}=\int_0^x w(u)\phi_1(x-u)du, \tag{118}$$

$$L^{-1}\frac{p}{(p^4+4a^4)}=\int_0^x w(u)\phi_2(x-u)du. \tag{119}$$

From (116) and (117) we thus obtain the deflection and slope to be

$$y=\frac{1}{EI}\int_0^x w(u)\phi_1(x-u)du+y_2\phi_2(x)+y_3\phi_1(x), \tag{120}$$

$$\frac{dy}{dx}=\frac{1}{EI}\int_0^x w(u)\phi_2(x-u)du+y_2\phi_3(x)+y_3\phi_2(x). \tag{121}$$

If we let

$$A=\frac{1}{EI}\int_0^s w(u)\phi_1(s-u)du, \qquad (122)$$

$$B=\frac{1}{EI}\int_0^s w(u)\phi_2(s-u)du, \qquad (123)$$

and make use of the fact that the slope and deflection both vanish at $x=s$, we obtain the following values for the constants y_2 and y_3:

$$y_2=\frac{B\phi_1(s)-A\phi_2(s)}{\phi_2^2(s)-\phi_1(s)\phi_3(s)}, \qquad (124)$$

$$y_3=\frac{B\phi_2(s)-A\phi_3(s)}{\phi_1(s)\phi_3(s)-\phi_2^2(s)}. \qquad (125)$$

If these values of y_2 and y_3 are substituted into (120), the value of the deflection is obtained.

VI. GENERAL CONSIDERATIONS

The above examples show the utility of the operational or Laplacian transform method of solving the differential equations which occur in the theory of structures.

The examples given are typical and other types of boundary conditions may be solved in the same manner. The advantages of the operational method are that one may write one equation for the whole span and also the work of determining the arbitrary constants of the solution is considerably reduced.

APPENDIX A. BASIC THEOREMS OF THE LAPLACIAN TRANSFORMATION

The utility of the Laplacian transformation in the solution of differential equations is based on some important relations which follow as a consequence of the fundamental Eq. (1). The most important of these theorems will be listed here for reference. They are established in references 2–4.

If $Lh(x)=g(p)$ then

$$Lh(sx)=g(p/s) \qquad \text{I.}$$

where s is a constant $s>0$.

$$L(dh/dx)=pg(p)-ph(0), \qquad \text{II.}$$

where $h(0)$ is the value of $h(t)$ at $t=0$.

$$L\left(\frac{d^nh}{dx^n}\right)=p^ng(p)-\sum_{k=0}^{k=n-1}h^{[k]}(0)p^{(n-k)}, \qquad \text{III.}$$

where $h^{[k]}(0)$ denotes the value of the kth derivative of $h(x)$ evaluated at $x=0$.

$$L^{-1}\epsilon^{-ap}g(p)=\begin{cases}0 & x<a\\ h(x-a) & x>0\end{cases}, \qquad \text{IV.}$$

where a is a positive constant.

If $L^{-1}g_1(p)=h_1(x)$ and $L^{-1}g_2(p)=h_2(x)$, then

$$L^{-1}\frac{g_1(p)g_2(p)}{p}=\begin{cases}\int_0^x h_1(u)h_2(x-u)du\\ \int_0^x h_2(u)h_1(x-u)du\end{cases}. \qquad \text{V.}$$

This theorem is known as the Faltung theorem.

APPENDIX B. TABLE OF LAPLACE TRANSFORMS

$$g(p)=p\int_0^\infty \epsilon^{-px}h(x)dx$$

No.	$g(p)$	$h(x)$
1.	p	$\delta(x)$, the Dirac impulse function $\delta(x)=\infty$ for $x=0$, $\delta(x)=0$, $x\neq 0$ $\int_{-\infty}^{+\infty}\delta(x)dx=1$
2.	$\frac{1}{p^n}$ $n=1, 2, 3, \cdots$	$\frac{x^n}{n!}$ (n a positive integer)
3.	$\frac{p^2}{(p^2+a^2)}$	$\cos(ax)$
4.	$\frac{pa}{(p^2+a^2)}$	$\sin(ax)$
5.	$\frac{p^2}{(p^2-a^2)}$	$\cosh(ax)$
6.	$\frac{pa}{(p^2-a^2)}$	$\sinh(ax)$
7.	$\frac{1}{(p^2-a^2)}$	$\frac{1}{a^2}\{\cosh(ax)-1\}$
8.	$\frac{p}{(p^4+4a^4)}$	$\frac{1}{4a^3}(\sin ax\cosh ax-\cos ax\sinh ax)=\phi_1(x)$
9.	$\frac{p^2}{(p^4+4a^4)}$	$\frac{1}{2a^2}(\sin ax\sinh ax)=\phi_2(x)$
10.	$\frac{p^3}{(p^4+4a^4)}$	$\frac{1}{2a}(\sin ax\cosh ax+\cos ax\sinh ax)=\phi_3(x)$
11.	$\frac{p^4}{(p^4+4a^4)}$	$\cos ax\cosh ax=\phi_4(x)$
12.	$\frac{p}{(p^4-a^4)}$	$\frac{1}{2a^3}(\sinh ax-\sin ax)$
13.	$\frac{p^2}{(p^4-a^4)}$	$\frac{1}{2a^2}(\cosh ax-\cos ax)$
14.	$\frac{p^3}{(p^4-a^4)}$	$\frac{1}{2a}(\sinh ax+\sin ax)$
15.	$\frac{p^4}{(p^4-a^4)}$	$\frac{1}{2}(\cosh ax+\cos ax)$

Radioactive Mercury in Mercury-Vapor Measurements

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(Received June 25, 1943)

Radioactive mercury, Hg^{197}, was prepared by deuteron bombardment of gold. This isotope was used as a tracer to establish an average concentration of 0.01 mg Hg per m^3 of air in a particular industrial operation suspected of causing chronic mercurialism. Mercury in quantities of 10^{-8} g was detected, and the method offers possibilities of higher sensitivity and more general application.

INTRODUCTION

IN an investigation of a particular industrial operation suspected of causing chronic mercurialism, it was necessary to measure minute quantities of mercury vapor under conditions where optical measurements[1,2] were not feasible. However, it was found that the measurements could be made by the use of radioactive mercury[3] as a tracer. Quantities of mercury vapor as small as 10^{-2} μg were determined in 2 liters of air. This concentration limit was adequate for the problem at hand, but the sensitivity could be increased by a factor of 10 to 100 if necessary.

The operation in question consisted of sealing small side-arm tubes (see left side of Fig. 1) to fluorescent lamp bulbs. The side arm contained about 30 mg of mercury. An operator heated the constricted end of the side-arm tube and a corresponding tubulation on the fluorescent bulb pressed the softened ends together, and blew gently into a mouth piece (not shown) connected by a rubber tube to the cotton filter. This operation was repeated 50 to 60 times an hour. Because of the appreciable vapor pressure of mercury at room temperature, this procedure involved the inhalation of some mercury vapor. Measurements were made to determine if this quantity exceeded the toxic limit of 0.1 mg Hg per cubic meter of air.[4]

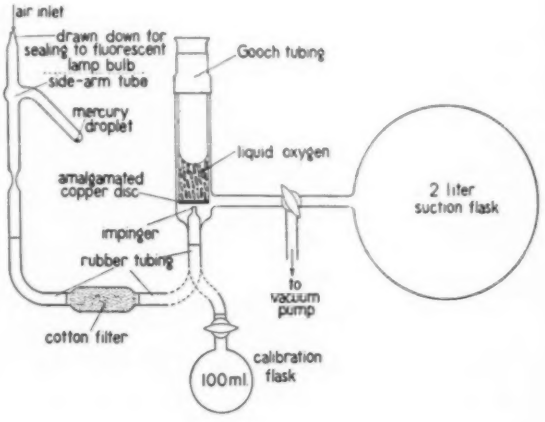

FIG. 1. Diagram of apparatus used in collecting mercury from air samples. A known volume of air is drawn through the side-arm tube upon connecting the evacuated suction flask to the impinger tube. The mercury vapor in the air is collected on the refrigerated, amalgamated copper disk. This disk is removed, covered with a thin lacquer film, and the content of radioactive mercury determined by beta-ray counting. The calibration flask was used in an attempt to obtain a known concentration of mercury vapor in air. This method was not satisfactory. Instead a standard solution was used.

APPARATUS AND METHOD

Figure 1 is a schematic diagram of the apparatus designed for these tests.[5] Radioactive mercury was placed in the side arm and air was drawn through the tube into an evacuated flask of 2.08-l capacity. Before reaching the flask, the mercury-laden air impinged on the amalgamated copper disk. This disk, fastened with rubber cement to the end of a tube filled with liquid oxygen, served to collect the radioactive mercury for measurement.

[1] M. Shepherd, S. Schukmann, R. H. Flinn, J. W. Hough, and P. A. Neal, J. Research Nat. Bur. Stand. **26**, 357 (1941).
[2] K. C. Clark and O. Oldenberg, J. Chem. Phys. **9**, 786 (1941).
[3] G. Friedlander and C. S. Wu, Phys. Rev. **63**, 227 (1943).
[4] This toxic limit, established by Shepherd *et al.* (reference 1), is based on a comprehensive experimental study and replaces the value of 0.25 mg per cubic meter proposed from a review of the literature by C. Goodman, Rev. Sci. Inst. **9**, 233 (1938).
[5] A more detailed description of the apparatus and industrial operations is appearing elsewhere. C. Goodman, J. W. Irvine, Jr., and C. F. Horan, J. Ind. Hyg. and Tox., in press (1943).

After the pressure in the apparatus reached atmospheric, the copper disk was removed, warmed to room temperature, dried, and covered with a thin film (1 mg/cm²) of lacquer. The activity of the disk was then measured on a thin-window, bell-type counter[6] with a recording counting-rate meter.[7]

An attempt was made to calibrate the apparatus by collecting the mercury vapor from a known volume (112 ml) of air in equilibrium with radioactive mercury. This was done by substituting a flask containing a small drop of mercury for the side-arm tube (Fig. 1). Although equilibrium was approached from both sides, the observed concentration always bracketed the true equilibrium value. These results indicated the need for more control to attain equilibrium than was considered necessary for this work. For purposes of quantifying the activity measurements in terms of weight of mercury, a standard solution of the radioactive mercury was prepared by dissolving 0.30 mg in 0.05 *M* nitric acid. An aliquot portion of this solution was evaporated on an amalgamated copper disk for measurement.

PREPARATION OF RADIOACTIVE MERCURY

Radioactive mercury, Hg^{197} (half-life = 23 hours)[3], was prepared by bombarding metallic gold with deuterons. A yield of 5.6 μc per μamp.h. was obtained by probe bombardment with 12-Mev deuterons in the M.I.T. cyclotron.

Five-mil gold foil was silver-soldered to a water-cooled probe and bombarded in the cyclotron chamber. Before the target was removed, a piece of 2-mil gold foil was placed over the active surface. This foil retained a large fraction of the radioactive mercury distilled from the target during the heating to melt the solder.

Both pieces of gold were amalgamated with 85 mg of inactive mercury carrier and placed in a simple vacuum still for purification of the radioactive mercury. The still consisted of a 100-mm length of 18-mm Pyrex tubing to which a 250-mm length of 3-mm capillary tubing was sealed. After introduction of the amalgamated gold, the open end of the larger tube was sealed and the still continuously evacuated with a Hyvac pump. Heat was applied to the large tube until the mercury distilled into the capillary and the tube collapsed. The capillary was then sealed between the gold and the condensed mercury. By gentle heating the mercury was distilled up the capillary in four stages, with a seal-off between each stage. It is necessary to carry out these distillations without allowing the mercury to boil. In this way mechanical transfer of traces of radioactive gold, also formed during the bombardment, is prevented. The final condensate was collected into a small droplet by centrifuging the last section of capillary.

EXPERIMENTAL RESULTS

Table I summarizes the experimental results. Measurements were made with a 14-mg and a 70-mg droplet of mercury in the side-arm tube and with the 70-mg droplet in the cotton filter, samples T-1 to T-9. Room-air blanks, B-1 and B-2, and calibration standards, S-1 to S-3, were measured. The time recorded in column 3 is the number of minutes between the collection of successive air samples. The samples were collected by allowing the evacuated flask to come to atmospheric pressure in 35 seconds to 1 minute. Activity measurements were made in duplicate and are corrected for decay of the activity and for counter background. The quantities of mercury are expressed in mg per cubic meter (m^3) and are calculated from an average specific activity of 9690 counts per minute (c.p.m.) per μg of mercury.

In samples T-1 to T-6, the first of each group shows a markedly higher activity than the succeeding two samples. This decrease is probably caused by the formation of an oxide film on the mercury while standing undisturbed and the lack of time for mercury vapor to diffuse into the straight portion of the side-arm tube after this has been freed of the vapor that accumulated during transfer of the droplets. With the droplet in the direct air stream, T-7 to T-9, the successive samples are more consistent.

The high blank values of the room air probably resulted from the fact that the apparatus was

[6] A. F. Kip and J. W. Irvine, Jr., Rev. Sci. Inst. (to be published, 1943).

[7] A. F. Kip, A. G. Bousquet, W. N. Tuttle, and R. D. Evans, Rev. Sci. Inst. (to be published, 1943).

TABLE I. Summary of numerical data from mercury tests.

Mercury data	Sample No.	Time in min.*	Counts per min. from 2.08 l of air	Hg in mg/m³ Observed	Hg in mg/m³ Average
14 mg in side arm of tube	T-1	50	655	0.034	0.036
			715	0.037	
	T-2	7	65	0.0034	0.0033
			61	0.0031	
	T-3	5	14	0.0007	0.0010
			26	0.0013	
70 mg in side arm of tube	T-4	5	367	0.019	0.019
			358	0.018	
	T-5	3	185	0.0095	0.0095
			186	0.0095	
	T-6	4	80	0.0041	0.0048
			105	0.0054	
70 mg in cotton filter	T-7	5	123	0.0063	0.011
			318	0.016	
	T-8	11	280	0.014	0.014
			284	0.014	
	T-9	11	204	0.011	0.011
			212	0.011	
Blank	B-1	—	32	0.0016	0.0018
			36	0.0019	
Blank	B-2	—	99	0.0051	0.0052
			101	0.0052	
				Counts per min. per 10^{-4} g Hg Observed	Counts per min. per 10^{-4} g Hg Average
5×10^{-4} mg Hg	S-1a		4600	9,200	
	S-1b	—	5330	10,660	
6.4×10^{-4} mg Hg	S-2	—	5850	9,160	9,690
13×10^{-4} mg Hg	S-3	—	12,440	9,750	

* This is the length of time the metallic mercury was in contact with the air in which the mercury vapor was measured.

set up in the radiochemical laboratory where the mercury was purified. Although the separation was made under a good fume hood, the mercury was divided and weighed in the room and sufficient mercury vaporized to contaminate the air.

Measurements of the activity on the outside of the liquid-oxygen tube after the collection of a very active sample (10,000 c.p.m.) from warm mercury were made. These results indicated that the removal of mercury vapor from the air by this apparatus was more than 99 percent complete.

The highest concentration of mercury vapor obtained from the side-arm tube was only 36 percent of the toxic limit of 0.1 mg per m³ set by Shepherd *et al.*[1] The average was about 10 percent of this value. From these results it may be concluded with some certainty that under the operating conditions in question toxic quantities of mercury vapor could not be inhaled.

OTHER APPLICATIONS

The experiments reported in this paper represent a highly specialized application of the use of radioactive mercury. However, the extreme sensitivity ($<10^{-8}$ g Hg) of the method suggests other applications where minute quantities of mercury are of interest. For instance, absorption of mercury through the skin, lungs, stomach, or teeth could be accurately measured. The rate of elimination of mercury from the body and factors which control that rate are subject to investigation by the radioactive tracer method. Other applications undoubtedly suggest themselves to the reader.